Inequality constraints and KKT

Detailed review with diagrams and worked solutions

Why an active constraint can have μ=0

Hessian, eigenvalues and curvature

The standard form used throughout

Minimize $f(x)$, subject to $g(x) \leq 0$.

$L(x,\mu)=f(x)+\mu g(x)$, with $\mu \geq 0$.

Upper bound $x \leq 2$: $g=x-2$.

Lower bound $x \geq 2$: $g=2-x$.

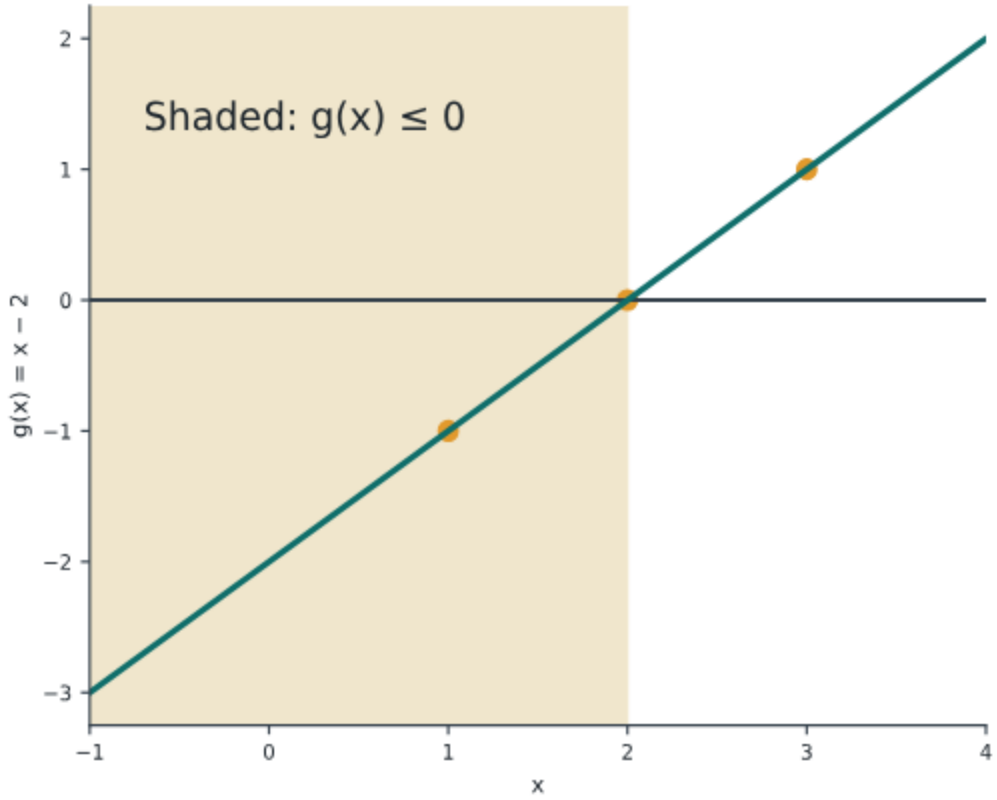
Feasibility is a test at a particular point

For $x \le 2$, evaluate $g(x) = x - 2$.

$x = 1$: $g = -1 < 0$, feasible and inactive.

$x = 2$: $g = 0$, feasible and active.

$x = 3$: $g = 1 > 0$, infeasible.



For feasible x, slack = $-g(x) = 2 - x$.

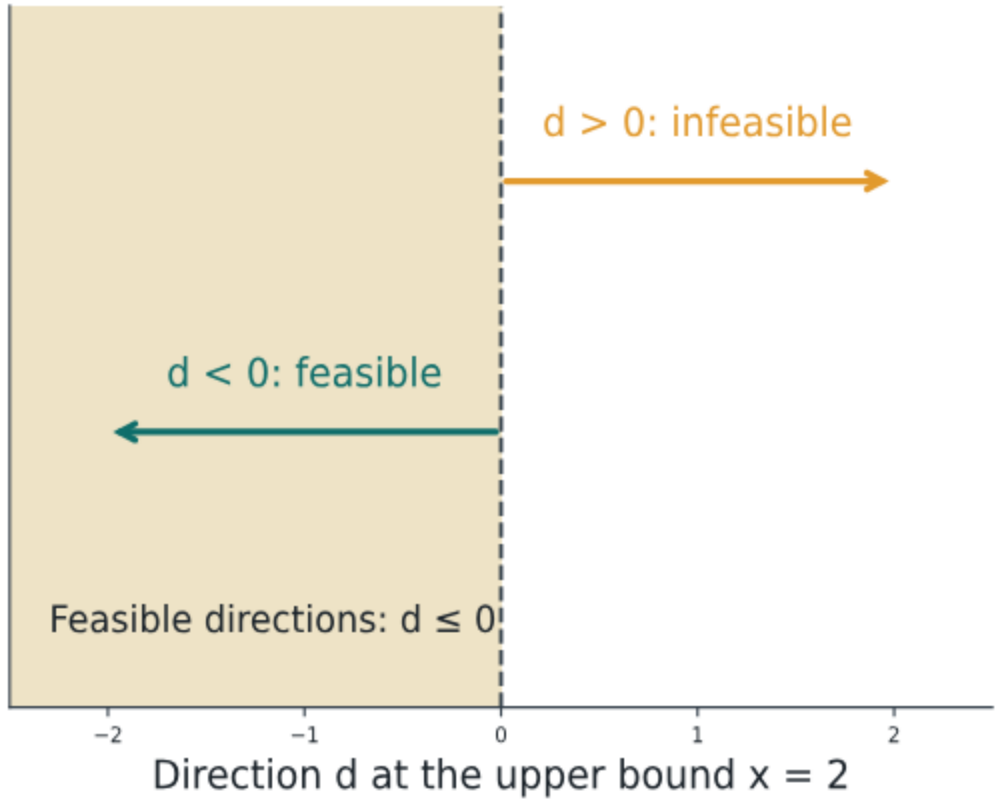
Feasible directions at an upper bound

At x=2, try a nearby point x+td, t>0.

Feasibility requires 2+td≤2.

Therefore d≤0: move left or stay still.

Any d>0 immediately violates the upper bound.



These directions are evaluated at the boundary x=2.

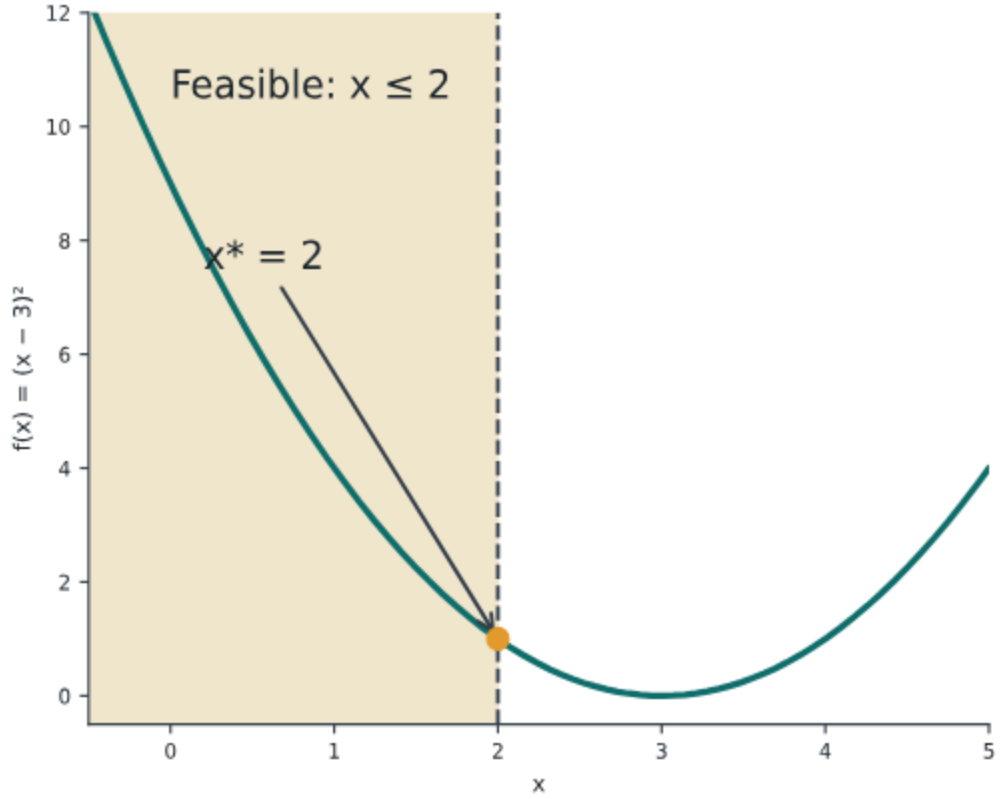
Objective descent can be blocked by a constraint

Let $f(x)=(x-3)^2$ with $x\le 2$.

At $x=2$, $f'(2)=-2$.

The first-order change is $f'(2)td=-2td$.

A decrease needs $d>0$, but feasibility needs $d\le 0$.



No feasible direction gives first-order descent at x=2.

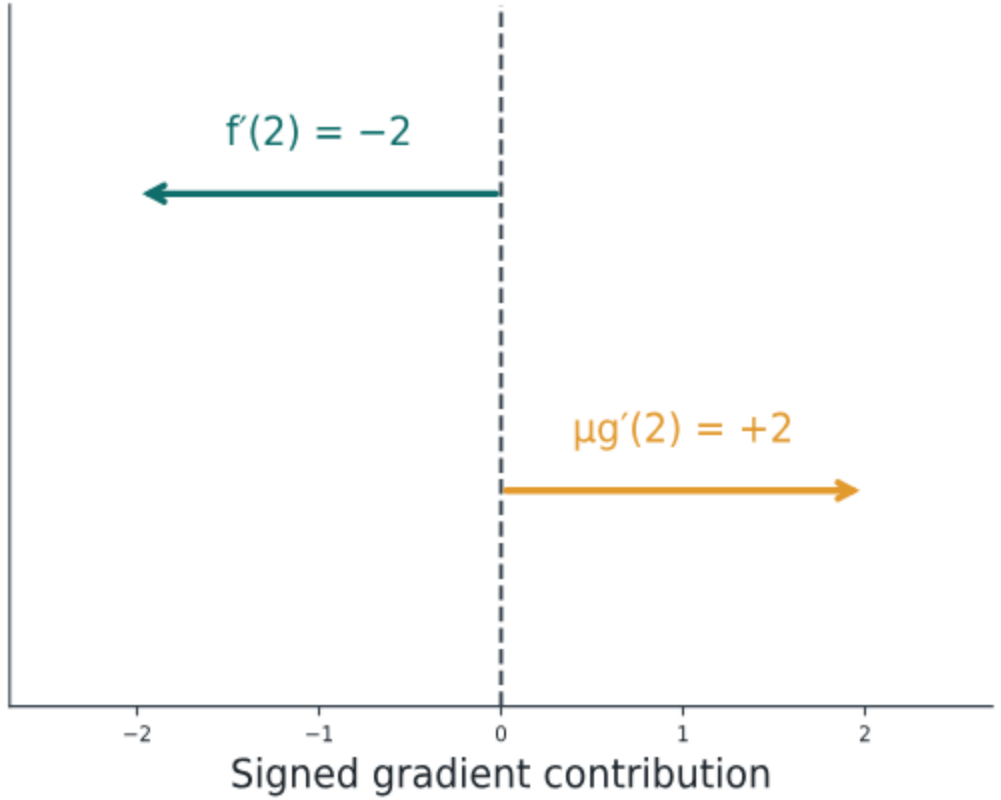
Lagrangian stationarity balances gradients

$L=(x-3)^2+\mu(x-2)$.

$\partial L/\partial x = 2(x-3)+\mu = 0$.

At $x=2$: $-2+\mu=0$, so $\mu=2$.

The two gradient contributions sum to zero.



Stationarity concerns L, not f alone.

Primal feasibility: is the candidate allowed?

Candidate $x=3$ gives $f=0$, the unconstrained minimum.

But $g(3)=3-2=1>0$, so reject $x=3$.

Candidate $x=2$ gives $g=0$ and is feasible.

A low objective value cannot excuse a violation.

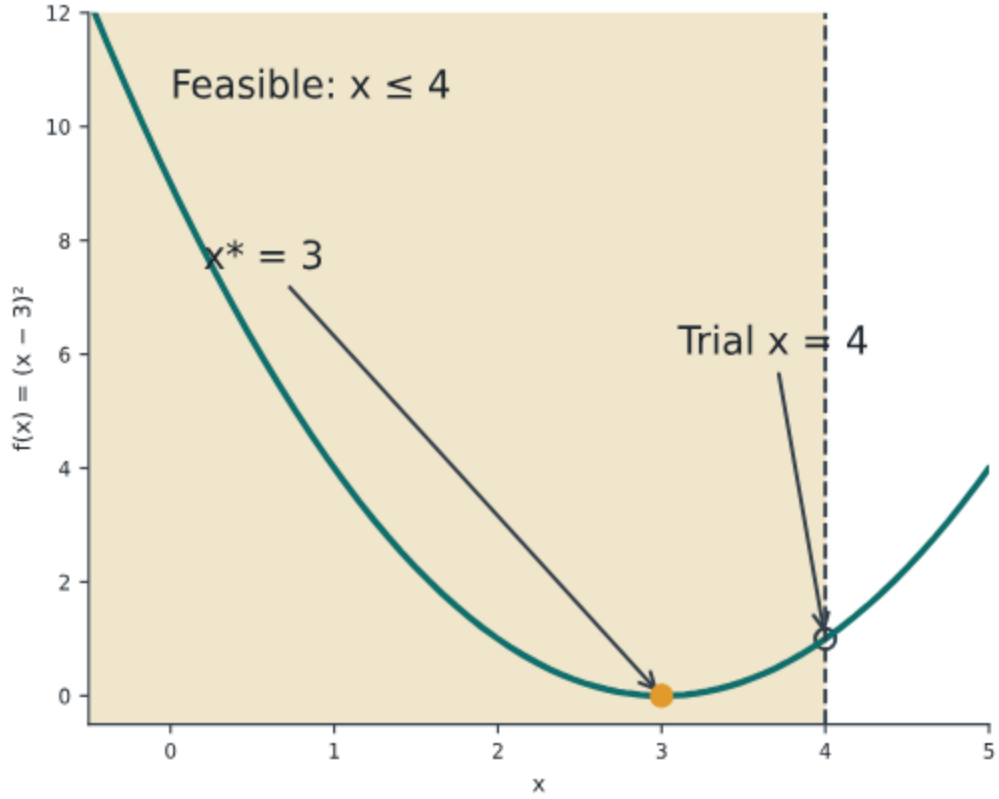
Dual feasibility rejects the wrong boundary

For min $(x-3)^2$ with $x \le 4$, try $x=4$.

Stationarity gives $2(4-3)+\mu=0$, so $\mu=-2$.

This violates $\mu \ge 0$. Reject the candidate.

Moving left from $x=4$ stays feasible and lowers f.



The true optimum is $x=3$, with $\mu=0$.

Complementary slackness is a product condition

$\mu \geq 0$, $g \leq 0$ and $\mu g = 0$ must all hold.

If $g < 0$, then μ must be zero.

If $\mu > 0$, then g must be zero.

If $g = 0$, the product is zero for every $\mu \geq 0$.

A question before the examples

An inequality constraint is active at an optimum.

Must its multiplier be positive?

A. Always positive B. Always zero

C. It can be zero or positive. Explain with an example.

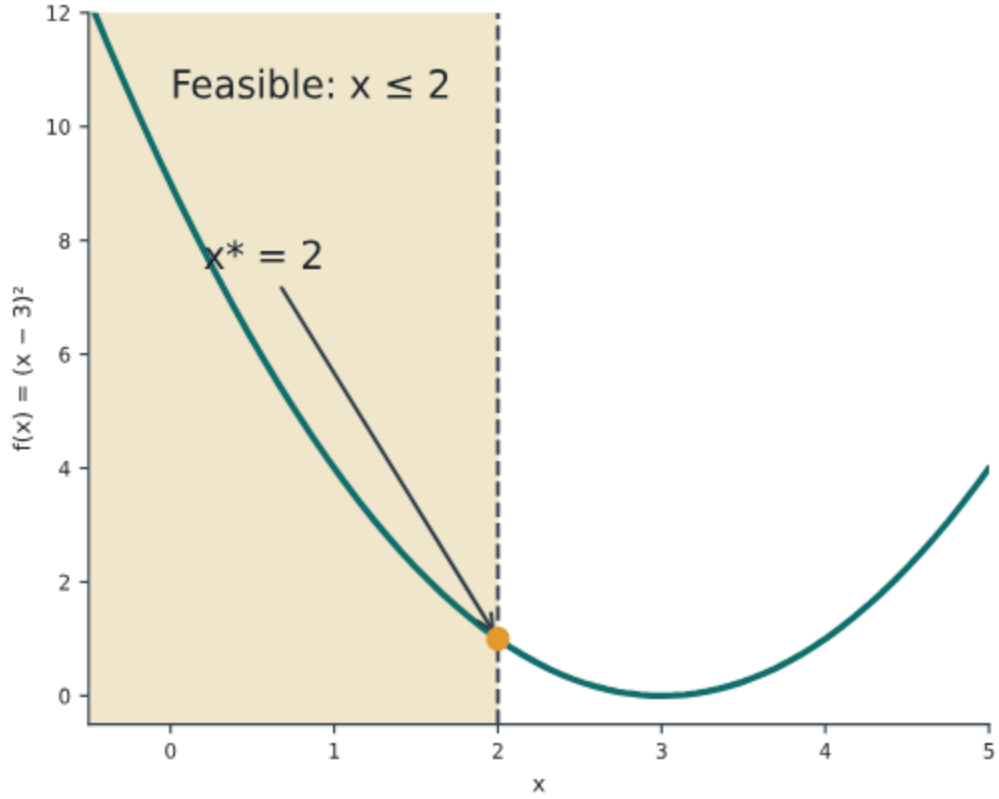
Example A: problem and geometric prediction

Minimize $(x-3)^2$ subject to $x \leq 2$.

The unconstrained minimum is $x=3$.

The feasible set ends before $x=3$.

Prediction: the optimal point lies at the bound $x=2$.



We will verify the prediction using both cases.

Example A: the inactive case

Assume $g = x - 2 < 0$.

Complementarity then requires $\mu = 0$.

Stationarity gives $2(x - 3) = 0$, hence $x = 3$.

But $g(3) = 1 > 0$. The inactive case is impossible.

Example A: the active case

Assume $g = x - 2 = 0$, so $x = 2$.

Stationarity: $2(2-3)+\mu = 0$.

Solve $-2+\mu = 0$ to obtain $\mu = 2$.

The candidate pair is $(x^*, \mu^*) = (2, 2)$.

Example A: all four checks and the conclusion

Primal: $g(2)=0\le 0$. Dual: $\mu=2\ge 0$.

Stationarity: $2(2-3)+2=0$.

Complementarity: $2\times 0=0$.

$f''=2>0$ and g is affine. Unique global minimum: $f(2)=1$.

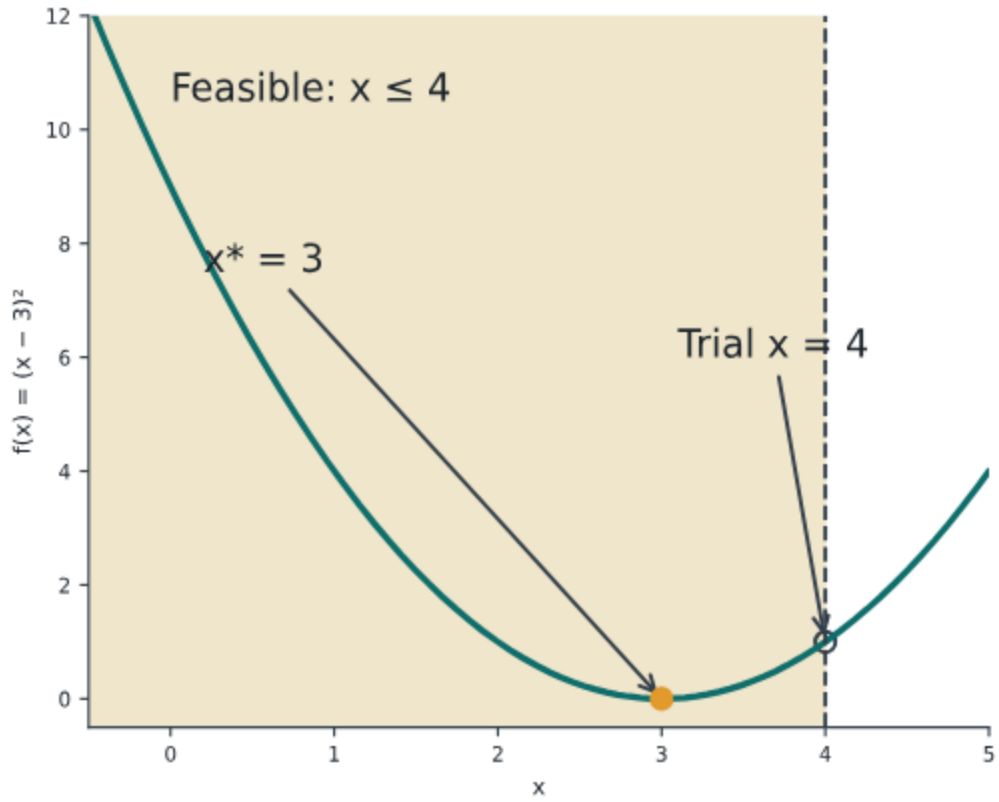
Example B: moving the bound to x≤4

The objective remains $f(x)=(x-3)^2$.

The point x=3 is now feasible.

At x=3, g=3−4=−1<0.

The bound is inactive and has slack 1.



A feasible minimum can lie strictly inside the feasible set.

Example B: calculation, checks and recap

$\mu(-1)=0$ gives $\mu=0$.

Stationarity: $2(3-3)+0=0$.

Primal: $g=-1\le 0$. Dual: $\mu=0\ge 0$. Complementarity: $0\times(-1)=0$.

Recap: $x^*=3$, inactive, $\mu^*=0$, unique global minimum $f^*=0$.

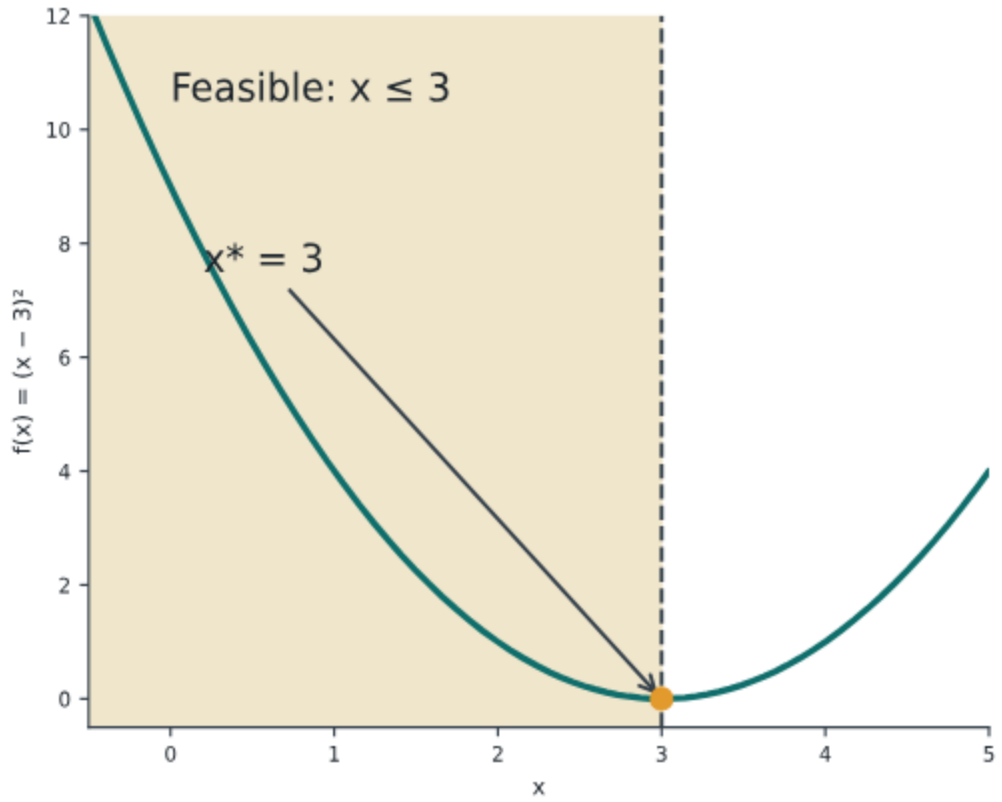
Example C: the bound passes through the minimum

Now impose $x \leq 3$.

The unconstrained minimum $x=3$ lies on the boundary.

$g(3)=3-3=0$, so the bound is active.

Both the objective minimum and the bound occur at $x=3$.



Active means equality at the point under consideration.

Example C: active with μ=0

$L=(x-3)^2+\mu(x-3)$.

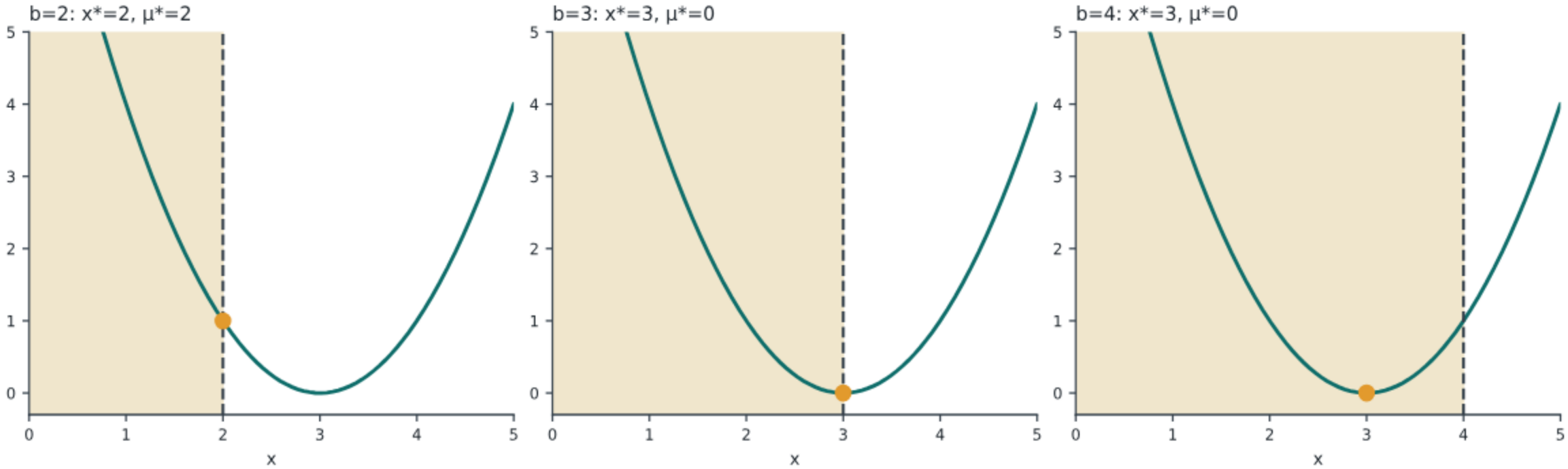
At $x=3$: stationarity gives $0+\mu=0$, hence $\mu=0$.

Primal: $g=0$. Dual: $\mu=0$. Complementarity: $0\times0=0$.

Recap: $x^*=3$ is active, $\mu^*=0$, and $f^*=0$ is the global minimum.

One objective with three positions of the bound

b=2: active, μ=2. b=3: active, μ=0. b=4: inactive, μ=0.



Answer: an active constraint can have a zero or a positive multiplier.

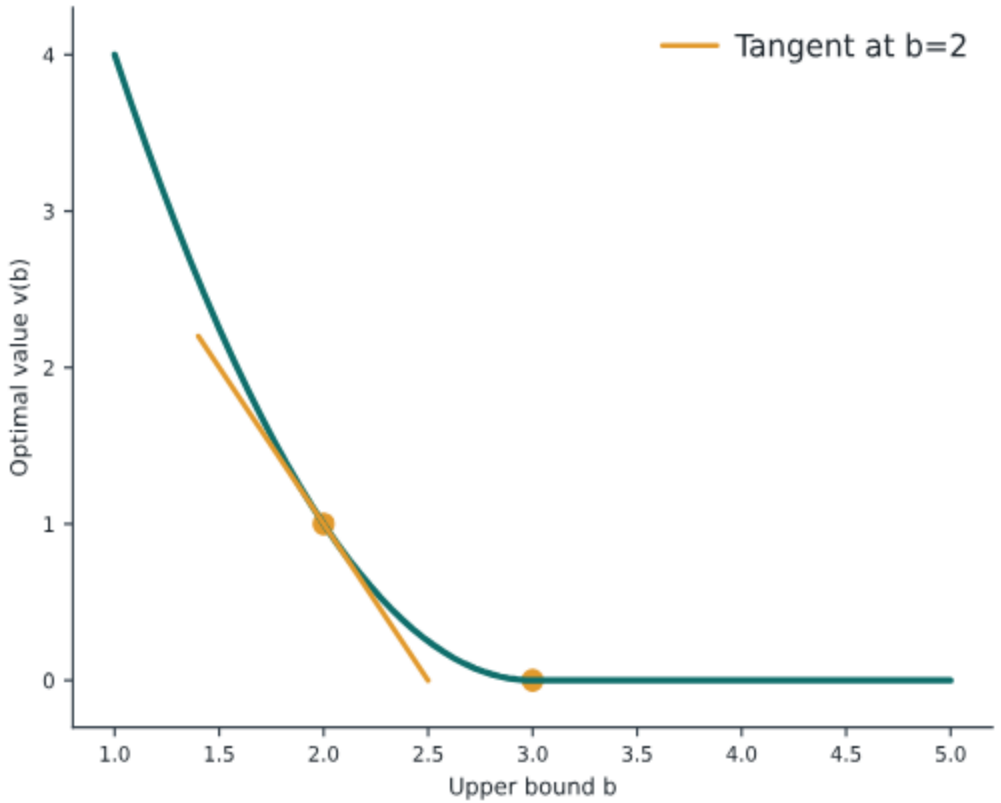
Sensitivity for this nonlinear example

Let $v(b)=\min\{(x-3)^2 : x\le b\}$.

$x^*(b)=\min(3,b)$.

$v(b)=(3-b)^2$ if $b<3$, and $v(b)=0$ if $b\ge 3$.

Here $\mu^*(b)=2\max(3-b,0)$, and $v'(b)=-\mu^*(b)$.



The tangent slope is a local rate of change.

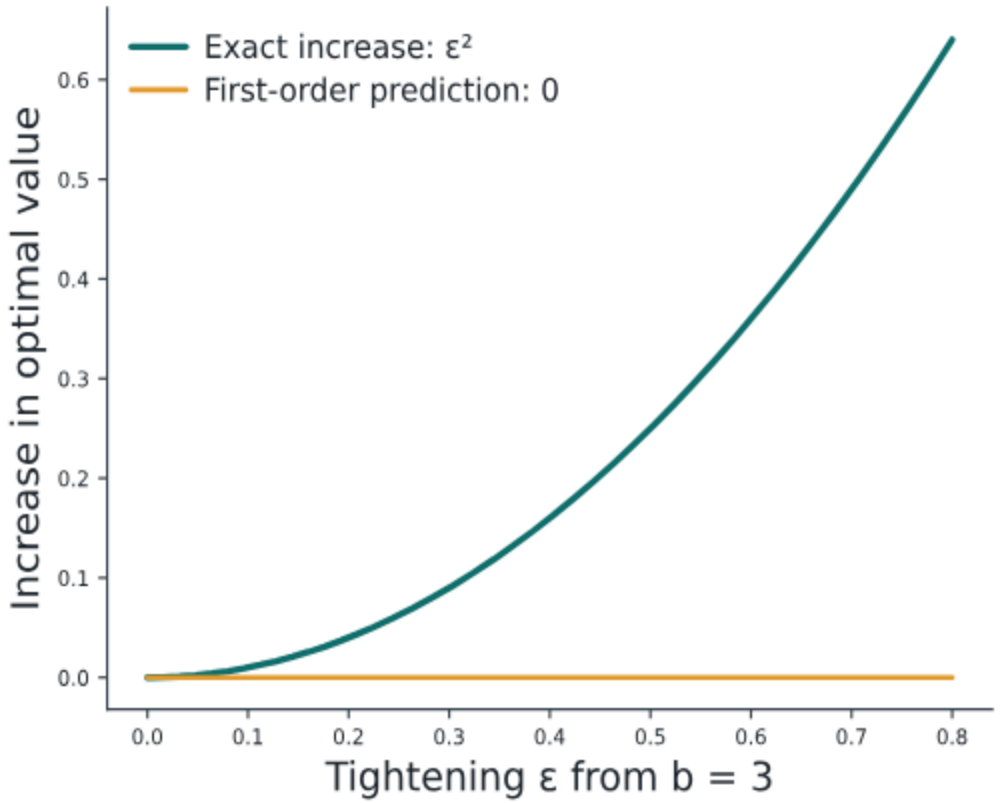
Zero first-order sensitivity can hide a finite change

At $b=3$, $\mu^*=0$ and $v'(3)=0$.

Tighten the bound to $b=3-\varepsilon$, where $\varepsilon>0$.

The new optimum is $x^*=3-\varepsilon$ and $v=\varepsilon^2$.

For $\varepsilon=0.5$, v increases from 0 to 0.25.



A zero derivative concerns the first-order term, not every finite change.

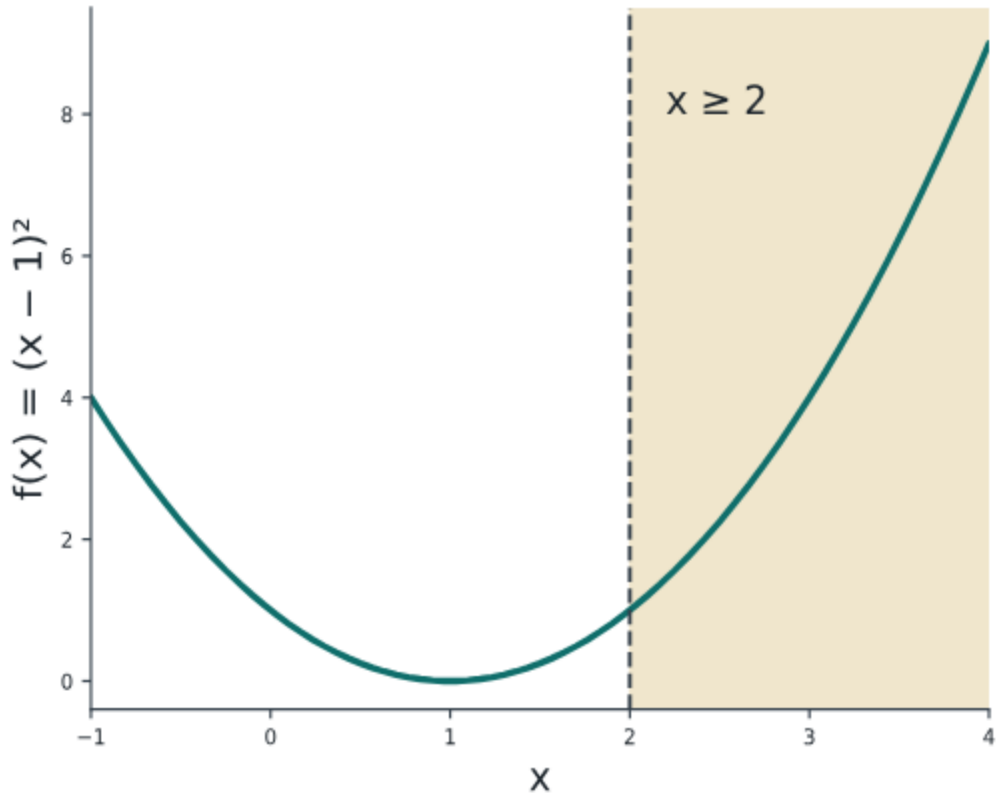
Lower-bound practice: predict before calculating

Minimize $f(x)=(x-1)^2$ subject to $x \geq 2$.

The unconstrained minimum is $x=1$.

Which x values are feasible?

Predict the optimum and write $g(x) \leq 0$.



The feasible region is to the right of the bound.

Lower-bound practice: the sign of g′

$x \geq 2$ becomes $g(x) = 2 - x \leq 0$.

$L = (x-1)^2 + \mu(2-x)$.

$g'(x) = -1$, so $\partial L / \partial x = 2(x-1) - \mu$.

Stationarity is $2(x-1) - \mu = 0$, with $\mu \geq 0$.

Lower-bound practice: both cases

Inactive: $\mu=0$ gives $x=1$. Reject because $x\geq 2$ fails.

Active: $g=0$ gives $x=2$.

Then $2(2-1)-\mu=0$ gives $\mu=2$.

The candidate is $(x^*,\mu^*)=(2,2)$.

Lower-bound practice: checks and recap

Primal: $2-2=0$. Dual: $\mu=2\geq 0$.

Stationarity: $2(2-1)-2=0$.

Complementarity: $2(2-2)=0$.

Strict convexity gives the unique global minimum: $x^*=2$, $f^*=1$.

Hessian: first check the stationary point

Let $f(x,y)=2x^2+2xy+2y^2$.

$\partial f/\partial x=4x+2y$ and $\partial f/\partial y=2x+4y$.

At (0,0), both first derivatives are zero.

This makes (0,0) a stationary point to classify.

Hessian: calculate each second derivative

$f_{xx} = 4, \quad f_{x\gamma} = 2, \quad f_{\gamma x} = 2, \quad f_{\gamma\gamma} = 4$

$$H = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} \qquad H - \lambda I = \begin{bmatrix} 4 - \lambda & 2 \\ 2 & 4 - \lambda \end{bmatrix}$$

Subtract λ only from the diagonal entries.

Eigenvalues: expand the determinant

$$\det(H - \lambda I) = \begin{vmatrix} 4 - \lambda & 2 \\ 2 & 4 - \lambda \end{vmatrix}$$

$$= (4 - \lambda)(4 - \lambda) - 2 \times 2$$

$$= 16 - 8\lambda + \lambda^2 - 4$$

$$= \lambda^2 - 8\lambda + 12$$

Eigenvalues satisfy $\det(H - \lambda I) = 0$.

Eigenvalues: factor and interpret

$\lambda^2-8\lambda+12=(\lambda-2)(\lambda-6)$.

Therefore $\lambda_1=2$ and $\lambda_2=6$.

Both are positive, so H is positive definite.

At the stationary point, this gives a strict minimum.

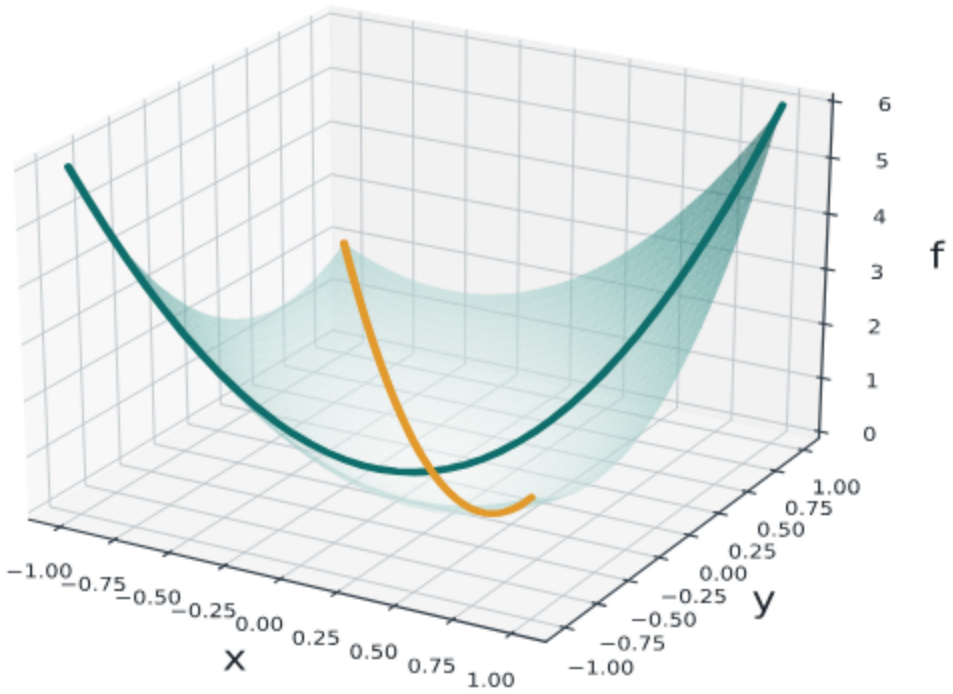
Positive curvature in three dimensions

$f=2x^2+2xy+2y^2$, with minimum at (0,0).

Along $y=x$: $f=6x^2$.

Along $y=-x$: $f=2x^2$.

The full Hessian confirms positive curvature in every nonzero direction.



Both highlighted sections rise away from the origin.

A saddle: calculate the Hessian and eigenvalues

$f(x,y) = x^2 + 4xy + y^2$

$\nabla f = (2x + 4y, 4x + 2y)$, so $\nabla f(0,0) = 0$.

$$H = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix} \qquad \det(H - \lambda I) = \begin{vmatrix} 2 - \lambda & 4 \\ 4 & 2 - \lambda \end{vmatrix}$$

$\det(H - \lambda I) = (2 - \lambda)^2 - 16 = \lambda^2 - 4\lambda - 12$

$(\lambda - 6)(\lambda + 2) = 0$, so $\lambda_1 = 6$ and $\lambda_2 = -2$.

A saddle: verify the directions explicitly

Along $y=x$: $f=x^2+4x^2+x^2=6x^2>0$ for $x\neq 0$.

Along $y=-x$: $f=x^2-4x^2+x^2=-2x^2<0$ for $x\neq 0$.

Along $x=0$: $f=y^2\geq 0$.

Opposite signs occur along the diagonals, not along $x=0$.

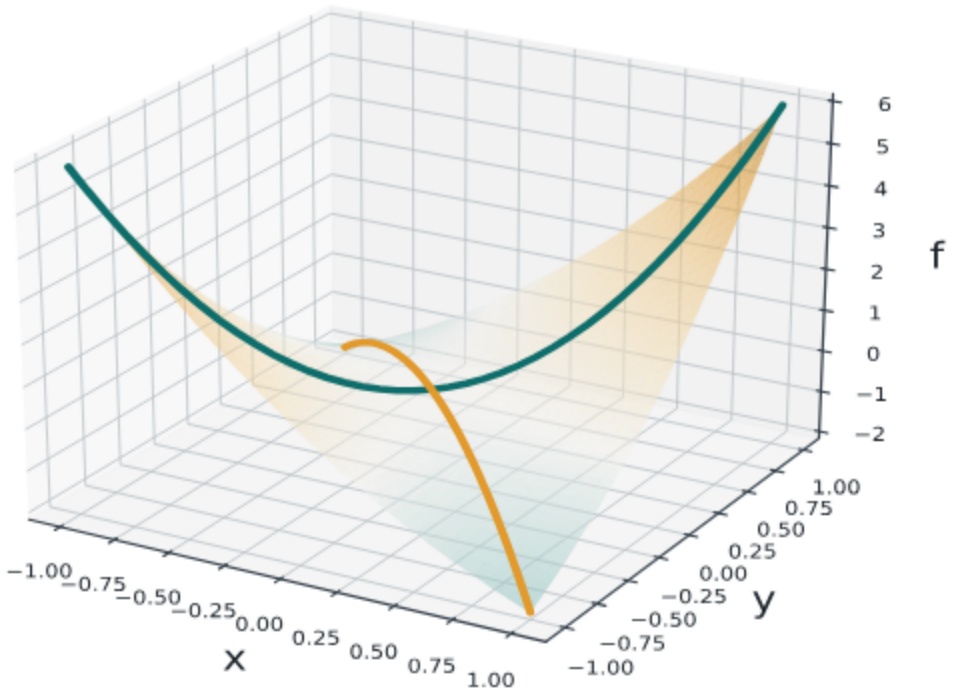
A saddle in three dimensions

$f=x^2+4xy+y^2$ at the origin.

The $y=x$ section rises.

The $y=-x$ section falls.

Every neighborhood contains values above and below $f(0,0)=0$.



The two highlighted sections have opposite curvature.

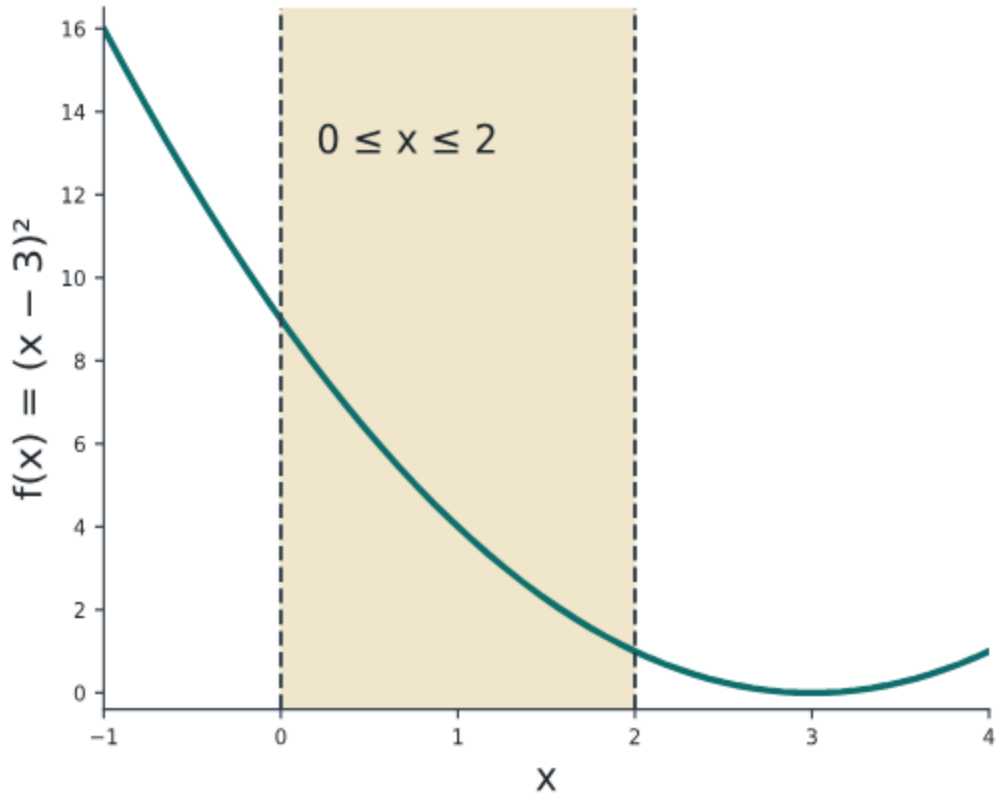
Practice: two inequality constraints

Minimize $(x-3)^2$ subject to $0 \le x \le 2$.

Write $g_1 = -x \le 0$ and $g_2 = x - 2 \le 0$.

Which constraint is active at the optimum?

Find x^*, μ_1^* and μ_2^*, then check all KKT conditions.



There is one nonnegative multiplier for each inequality.

Two constraints: Lagrangian and candidate cases

$L=(x-3)^2-\mu_1 x+\mu_2(x-2)$.

Stationarity: $2(x-3)-\mu_1+\mu_2=0$.

Complementarity: $\mu_1 x=0$ and $\mu_2(x-2)=0$.

Consider the interior $x\in(0,2)$ and the two endpoints.

Two constraints: reject two cases

Interior: both multipliers are zero. Stationarity gives x=3, outside [0,2].

At x=0, $g_2=-2<0$, so $\mu_2=0$.

Stationarity becomes $-6-\mu_1=0$, so $\mu_1=-6$.

Reject x=0 because $\mu_1\geq 0$ fails.

Two constraints: solve the upper endpoint

At $x=2$, $g_1=-2<0$, so $\mu_1=0$.

Stationarity: $-2-0+\mu_2=0$, hence $\mu_2=2$.

Primal: $(g_1,g_2)=(-2,0)$. Dual: $(\mu_1,\mu_2)=(0,2)$.

Complementarity: $0\times(-2)=0$ and $2\times0=0$.

Two constraints: short recap

$x^*=2$ and $f^*=1$.

The lower bound is inactive, with $\mu_1^*=0$.

The upper bound is active, with $\mu_2^*=2$.

Strictly convex f on the convex interval gives a unique global minimum.

Final questions before the answers

For min $(x-1)^2$, consider the following separately.

A. $x \geq 1$: find x^*, decide active/inactive, and find μ^*.

B. $x \geq 0$: find x^*, decide active/inactive, and find μ^*.

Does $\mu=0$ tell you whether the constraint is active?

Final answers and a reusable check

A: $g=1-x$. $x^*=1$ gives $g=0$ and $\mu^*=0$. Active.

B: $g=-x$. $x^*=1$ gives $g=-1$ and $\mu^*=0$. Inactive.

Both have $f^*=0$. The multiplier alone does not determine activity.

For each answer: check feasibility, activity, multiplier signs, stationarity and complementarity.