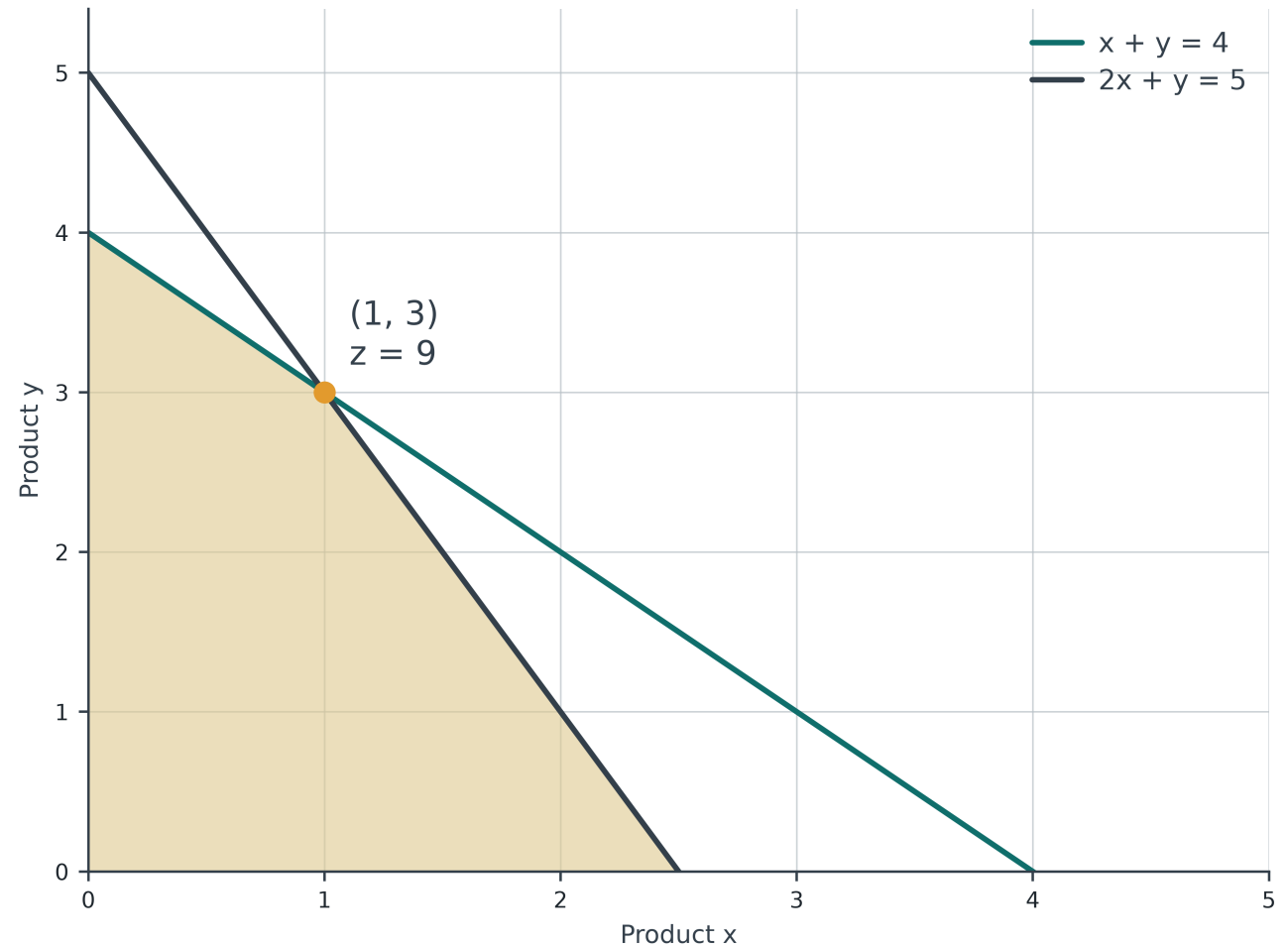


Linear programming and duality

2105623 Optimization of Chemical Processes

26 October 2026 · 180-minute class

Geometry, Simplex, optimality certificates, and resource values



Plan for the three-hour class

Minutes	Topic
0–30	LP geometry and corner solutions
30–70	Basic feasible solutions and Simplex
70–80	Break
80–125	Duality and complementary slackness
125–165	Shadow prices and sensitivity
165–180	Exit ticket, solutions, and recap

What makes a model linear?

$\max c^T x$ subject to $Ax \leq b, x \geq 0$

Coefficients are fixed; decision variables are continuous.

$x + 2y$ is linear. xy and x^2 are not.

A binary restriction changes an LP into a MILP.

Running example: two production routes

Item	Route x	Route y	Daily limit
Resource 1	1 unit/t	1 unit/t	4 units/day
Resource 2	2 units/t	1 unit/t	5 units/day
Contribution	$3 \times \$100/\text{t}$	$2 \times \$100/\text{t}$	Maximize

x, y = tonnes/day; contribution = \$100/day units.

Choose continuous production; no fixed startup cost.

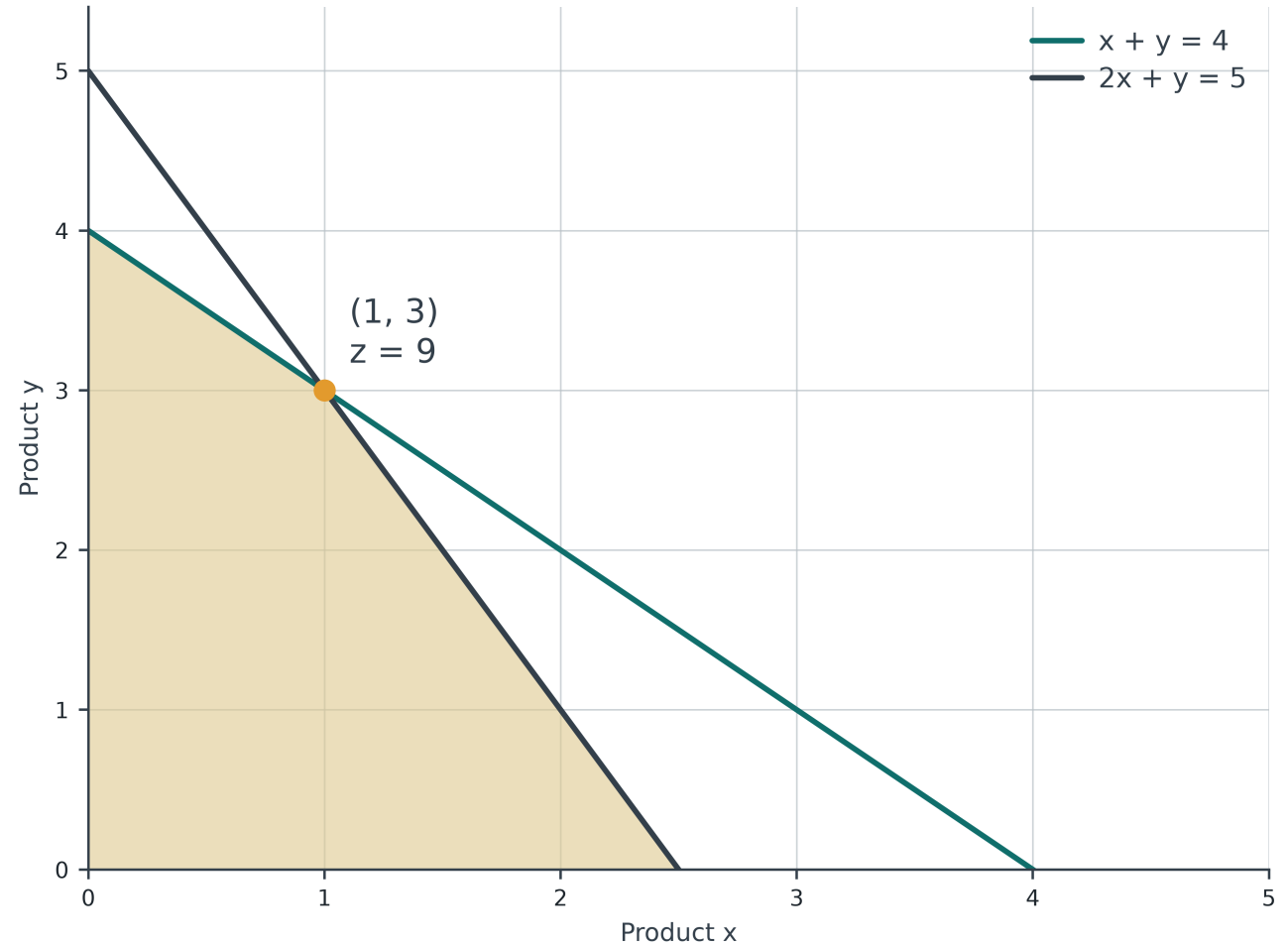
The production LP

$$\max z = 3x + 2y$$

$$x + y \leq 4$$

$$2x + y \leq 5$$

$$x \geq 0, y \geq 0$$



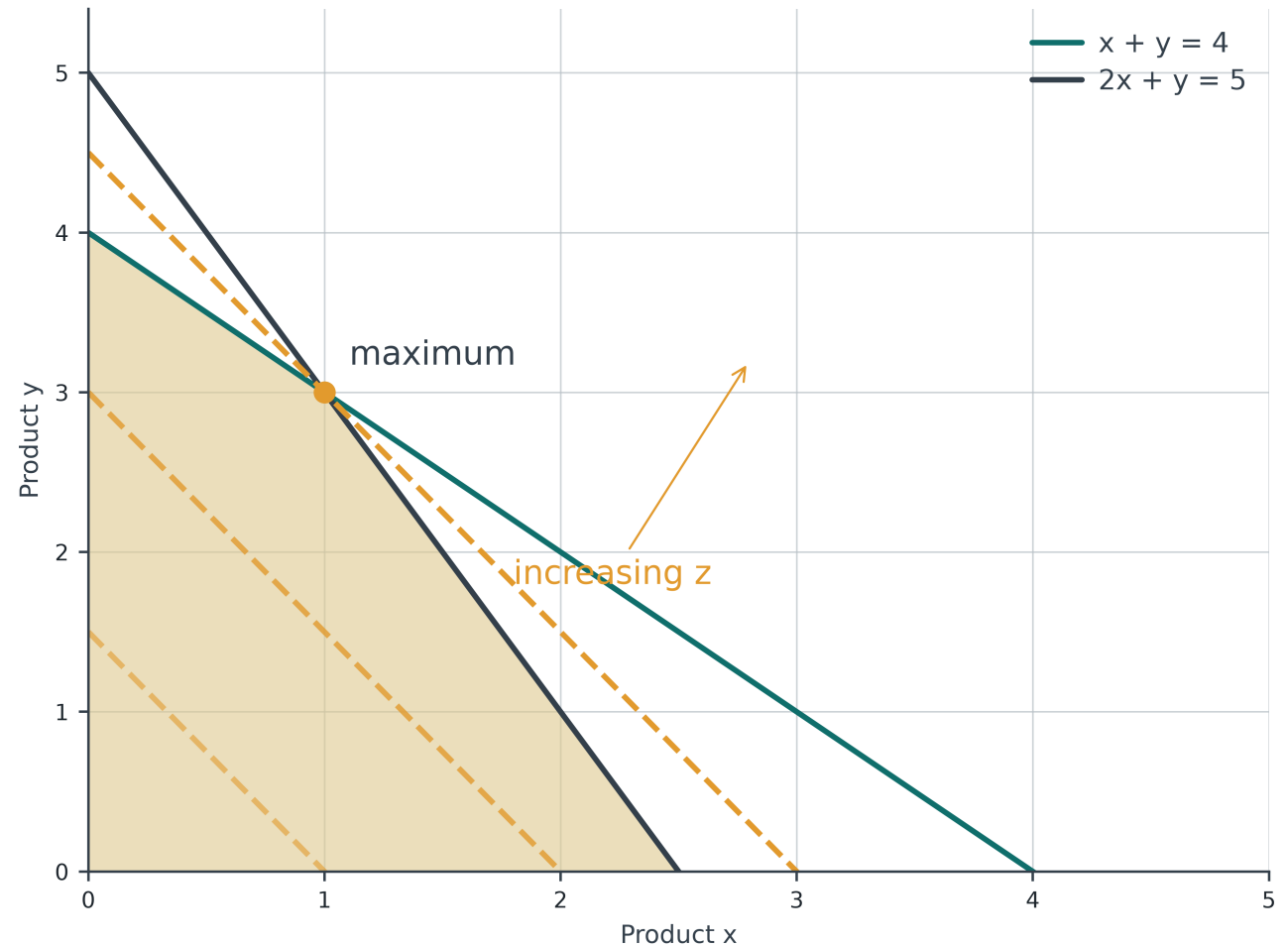
Objective contours and the best feasible point

$3x + 2y = z$ gives parallel lines.

Move the line toward larger z .

The last contact is $(x^*, y^*) = (1, 3)$.

$z^* = 9$, or \$900/day.



Why compare extreme points?

Vertex (x, y)	$z = 3x + 2y$
(0, 0)	0
(2.5, 0)	7.5
(1, 3)	9
(0, 4)	8

On a nonempty bounded polytope, a linear objective has an optimal extreme point.

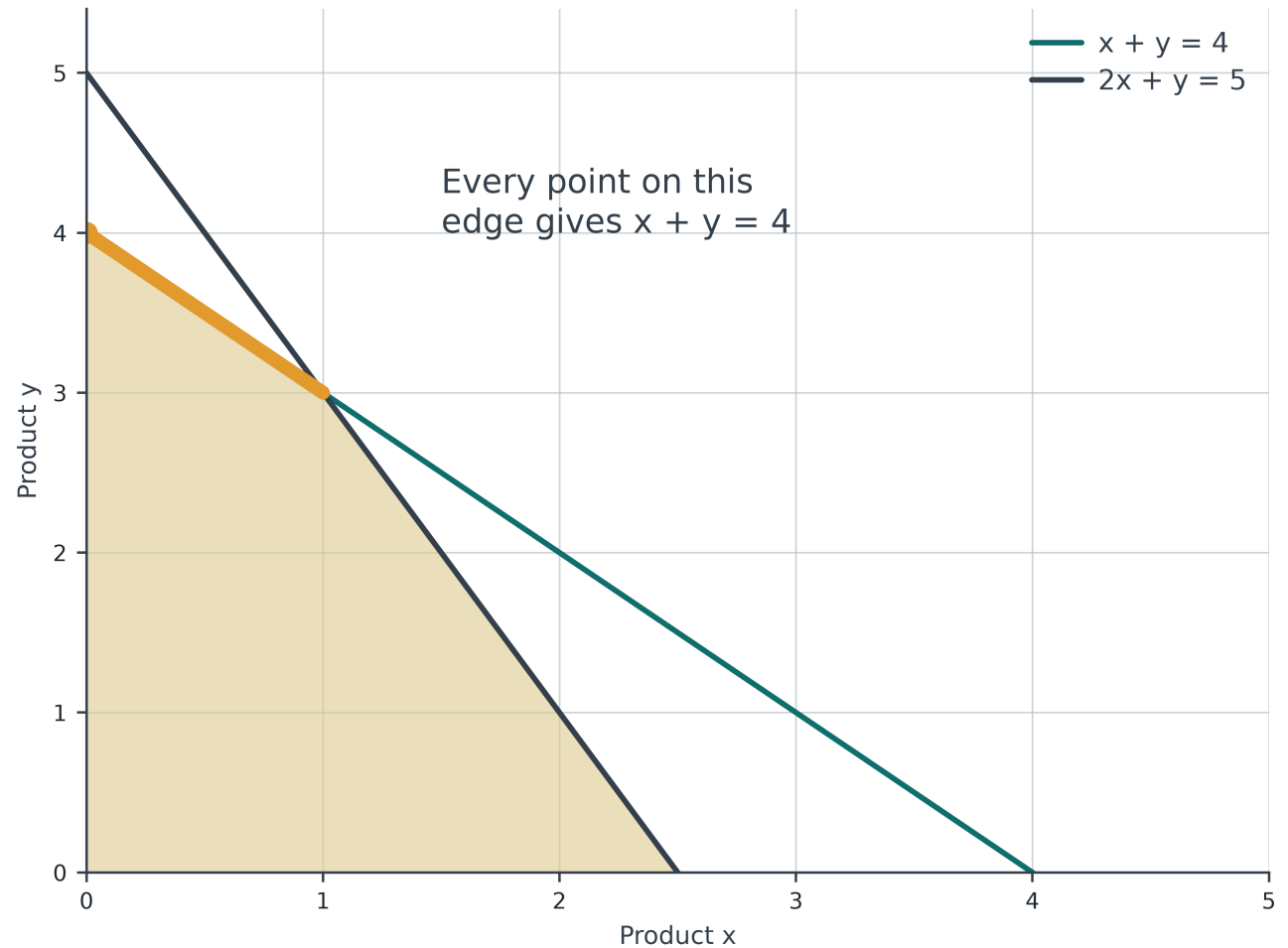
A convex combination cannot beat all vertices used in that combination.

An LP can have many optimal solutions

Change only the objective to $\max x + y$.

Every point from $(0,4)$ to $(1,3)$ is optimal.

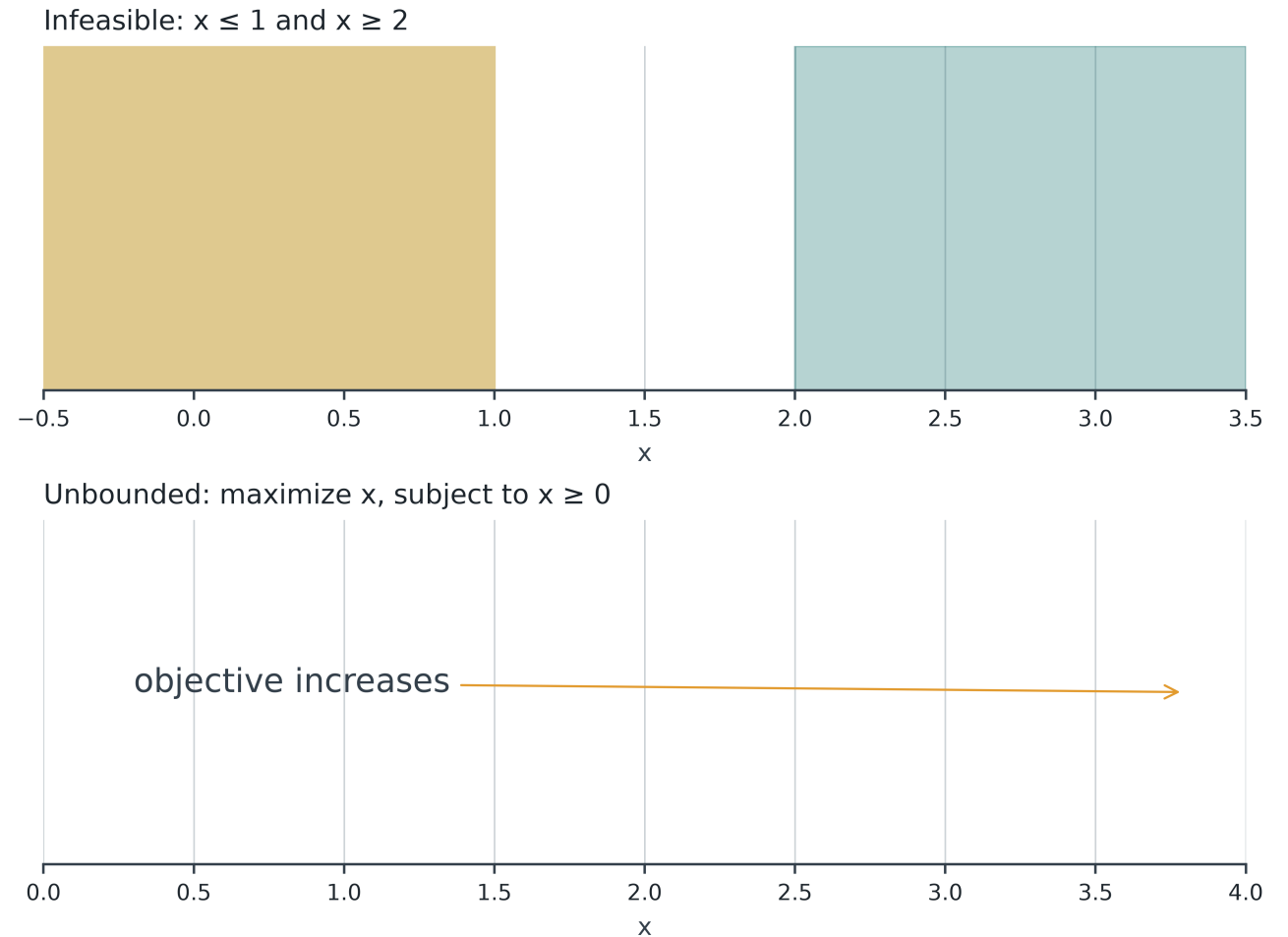
The entire optimal edge has value 4.



Infeasible and unbounded are different

Infeasible: $x \leq 1$ and $x \geq 2$ cannot both hold.

Unbounded: max x with only $x \geq 0$ has no finite maximum.



Quick check: feasibility before profit

Original LP: $\max 3x+2y$; $x+y \leq 4$, $2x+y \leq 5$, $x,y \geq 0$.

A: (2,2) gives $z=10$. B: (1,3) gives $z=9$.

Which point can be accepted? Explain the failed constraint.

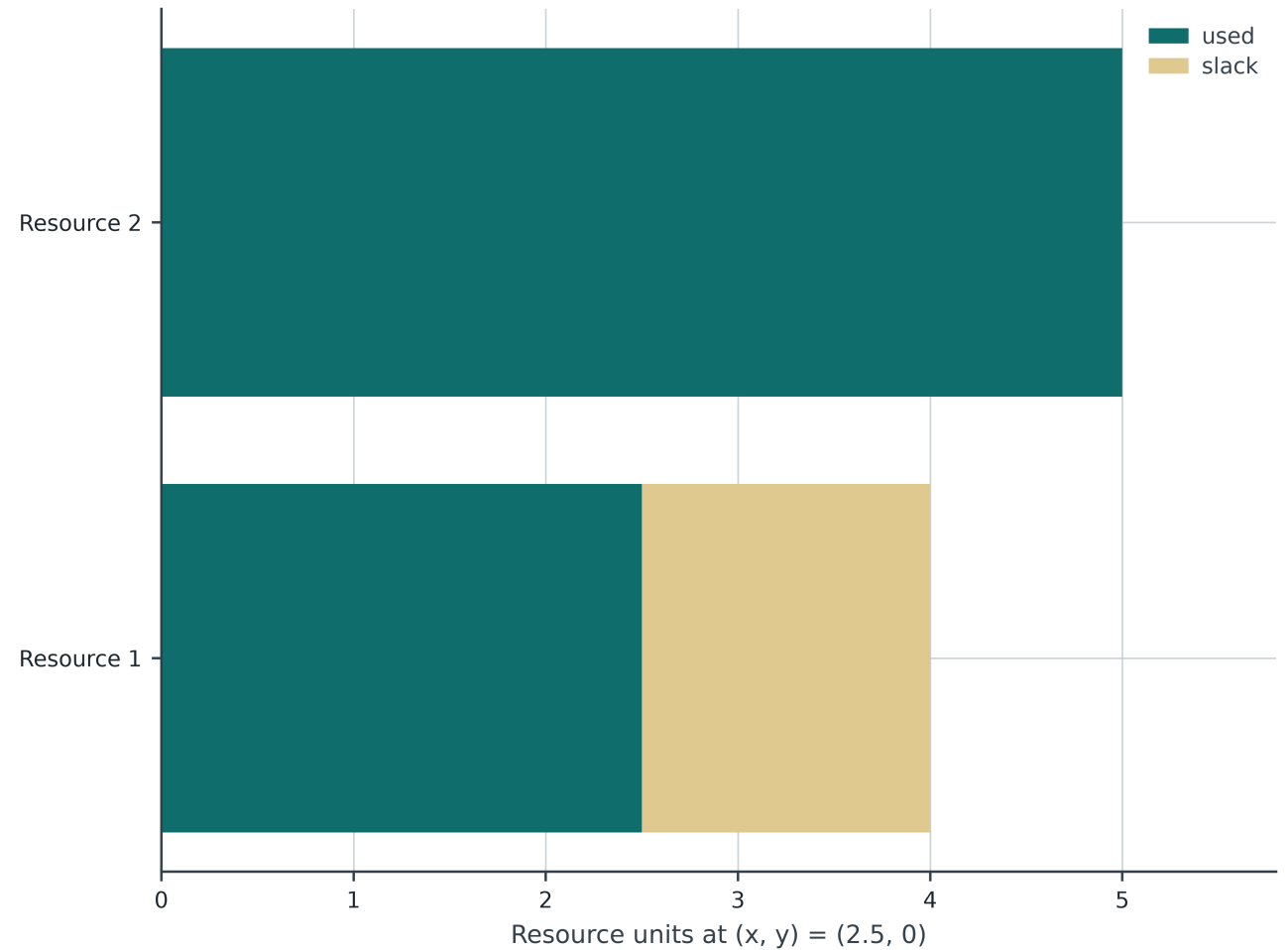
Slack variables and standard form

$$x + y + s_1 = 4$$

$$2x + y + s_2 = 5$$

$$x, y, s_1, s_2 \geq 0$$

s_1 and s_2 measure unused resources.



A basis selects the variables we solve for

Basis	Nonbasic variables	Basic solution
$\{s_1, s_2\}$	$x=y=0$	$s_1=4, s_2=5$
$\{x, s_1\}$	$y=s_2=0$	$x=2.5, s_1=1.5$
$\{x, y\}$	$s_1=s_2=0$	$x=1, y=3$

Two independent equations $\rightarrow$ choose two basic variables.

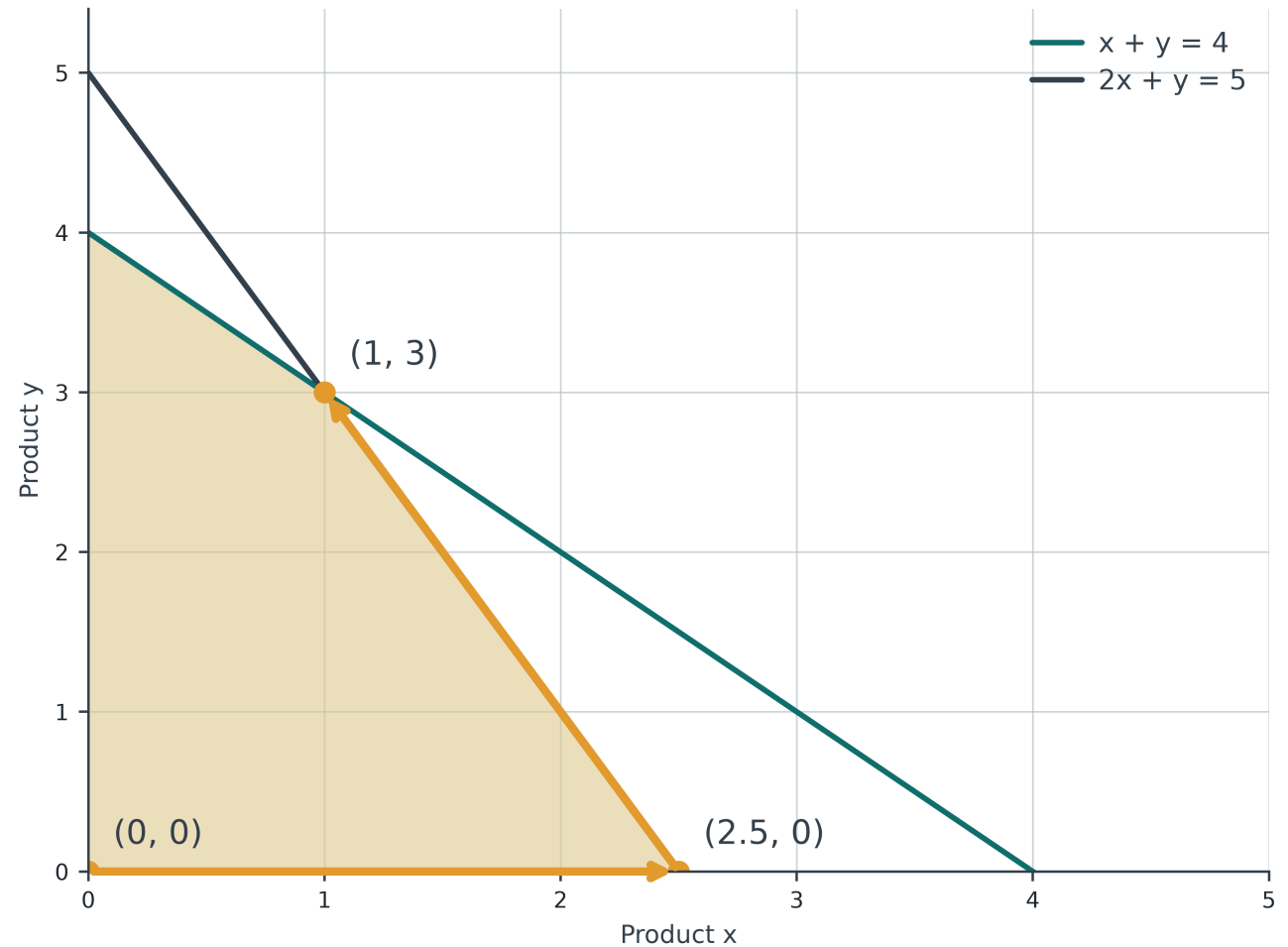
Set the other two variables to zero, then solve.

A basic solution is feasible only if every variable is nonnegative.

The Simplex decision at each step

1. Choose an improving nonbasic variable.
2. Increase it until a basic variable reaches zero.
3. Pivot: exchange the entering and leaving variables.

Positive reduced cost improves z if a positive feasible step exists.



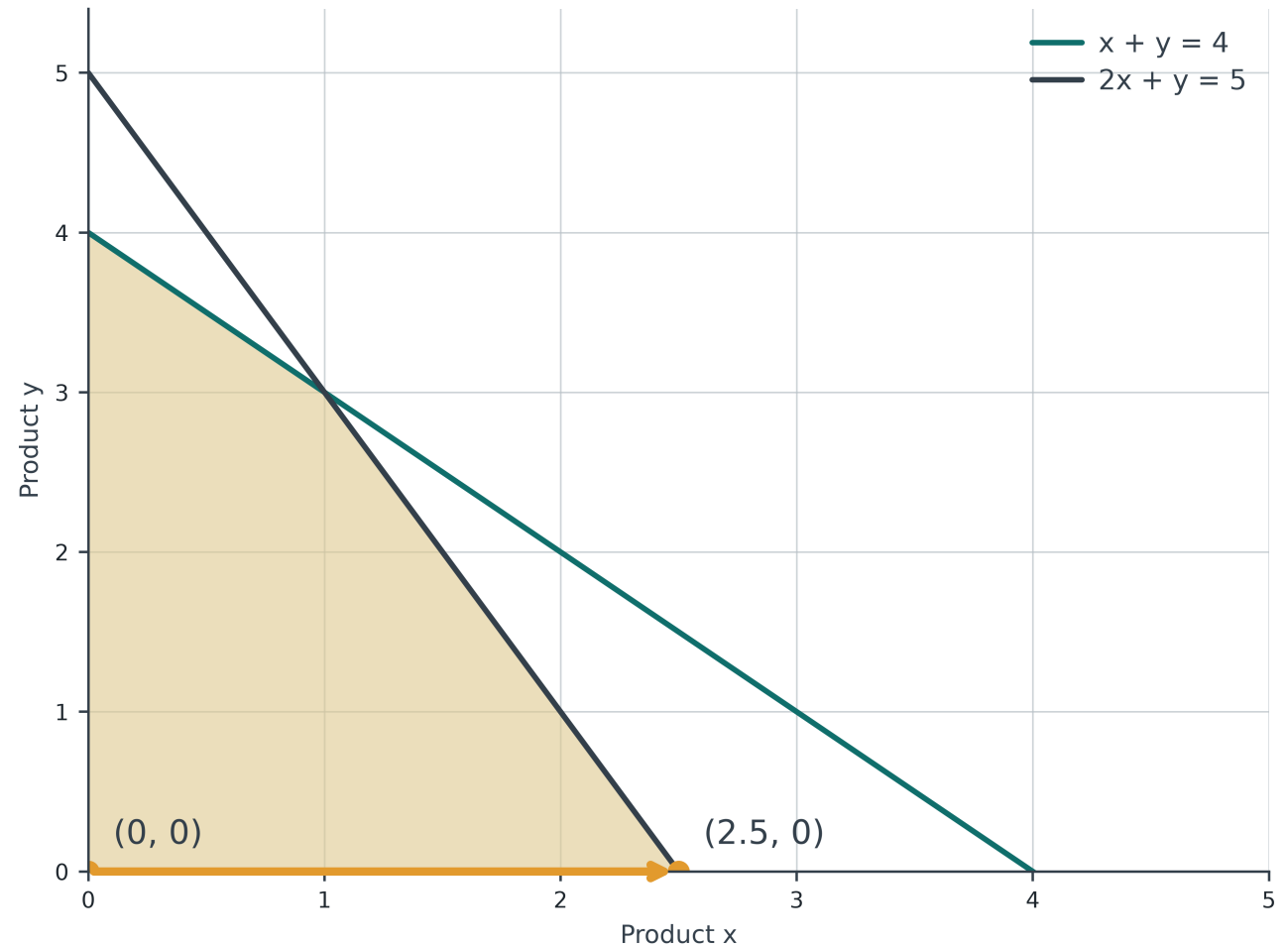
Pivot 1: x enters, s_2 leaves

At the origin: $z = 3x + 2y$. Choose x .

With $y=0$: $s_1=4-x$; $s_2=5-2x$.

Ratio test: $4/1=4$; $5/2=2.5$.

Move to $(2.5, 0)$; $z=7.5$.



Dictionary after the first pivot

$$x = 2.5 - 0.5y - 0.5s_2$$

$$s_1 = 1.5 - 0.5y + 0.5s_2$$

$$z = 7.5 + 0.5y - 1.5s_2$$

Nonbasic: $y=s_2=0$. The $+0.5$ coefficient means y can improve z .

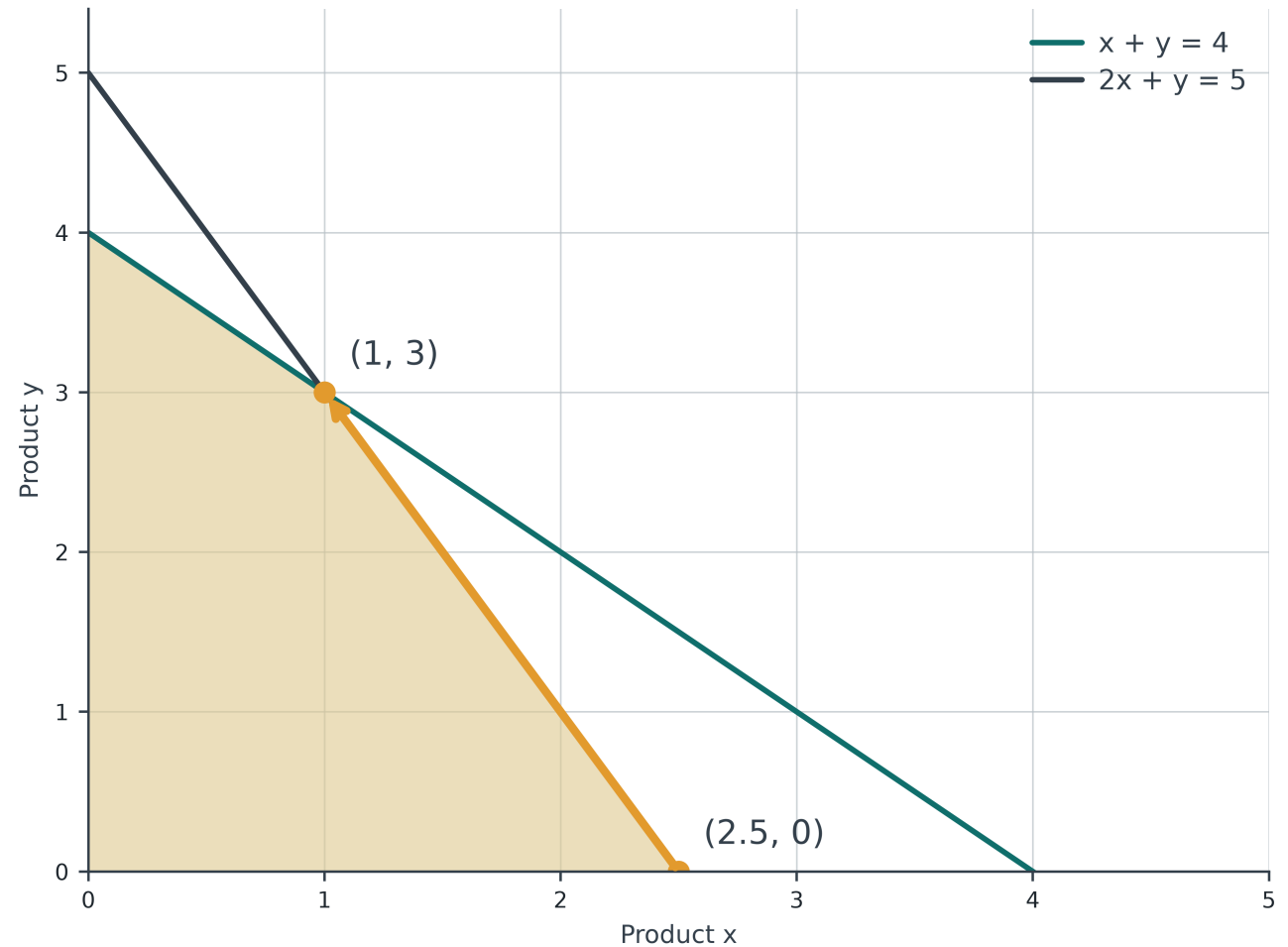
Pivot 2: y enters, s_1 leaves

Hold $s_2=0$ and increase y .

$x \geq 0$ gives $y \leq 2.5/0.5 = 5$.

$s_1 \geq 0$ gives $y \leq 1.5/0.5 = 3$.

Thus $y=3$, $x=1$, $s_1=s_2=0$; $z=9$.



The final dictionary certifies optimality

$$x = 1 + s_1 - s_2$$

$$y = 3 - 2s_1 + s_2$$

$$z = 9 - s_1 - s_2$$

Because $s_1, s_2 \geq 0$, every feasible point satisfies $z \leq 9$.

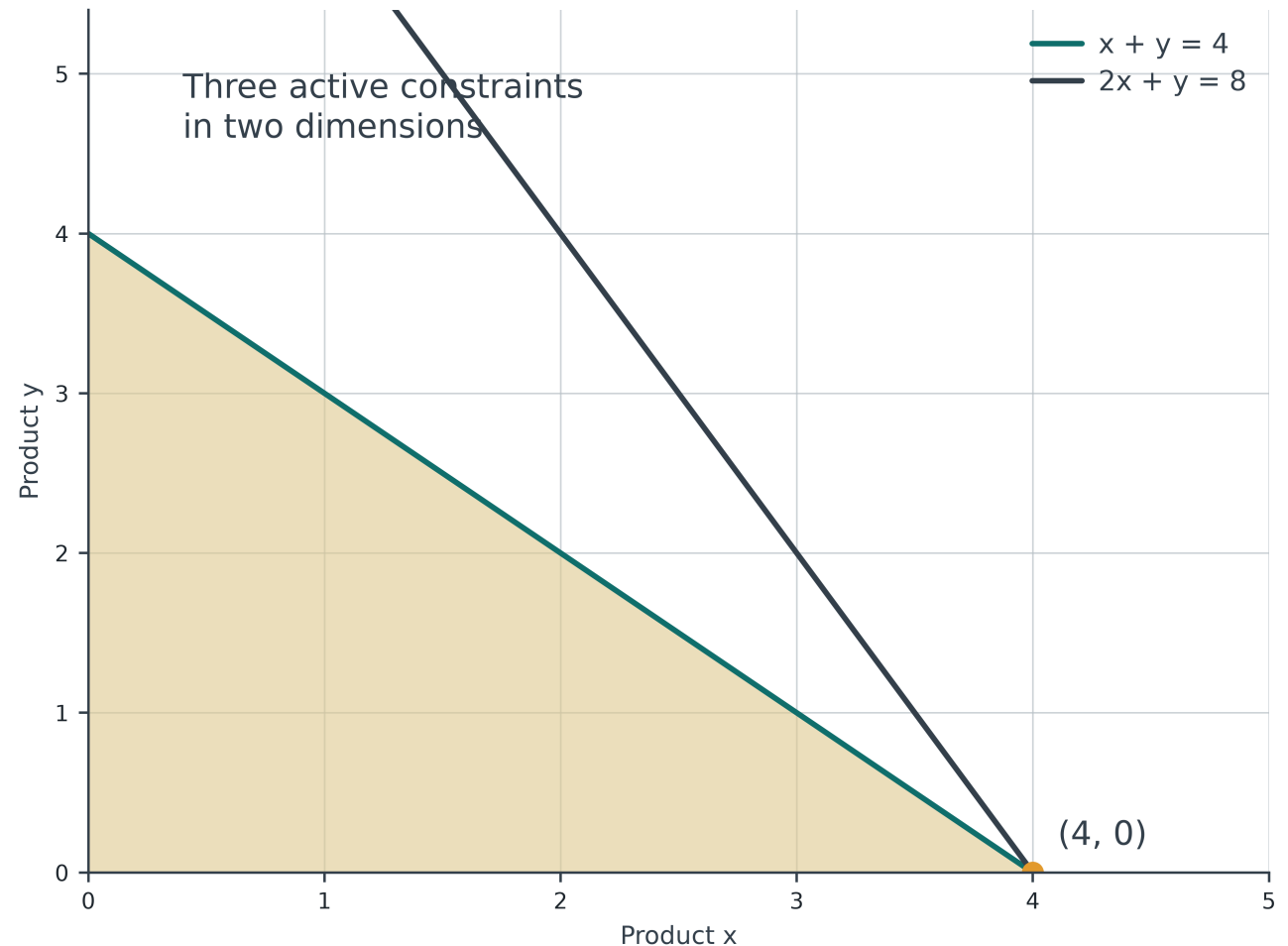
Degeneracy and starting feasibility

A degenerate BFS has at least one zero basic variable.

A pivot may change the basis without improving the point.

At $b=(4,8)$, the basis $\{x,y\}$ gives $x=4, y=0$.

If a feasible slack basis is unavailable, use Phase I or another initialization.



Practice: change the contribution

Keep $x+y \leq 4$, $2x+y \leq 5$, $x, y \geq 0$.

Change the objective to $\max z = x + 3y$.

Choose the first entering variable and apply the ratio test.

Predict the optimum and write a stopping argument.

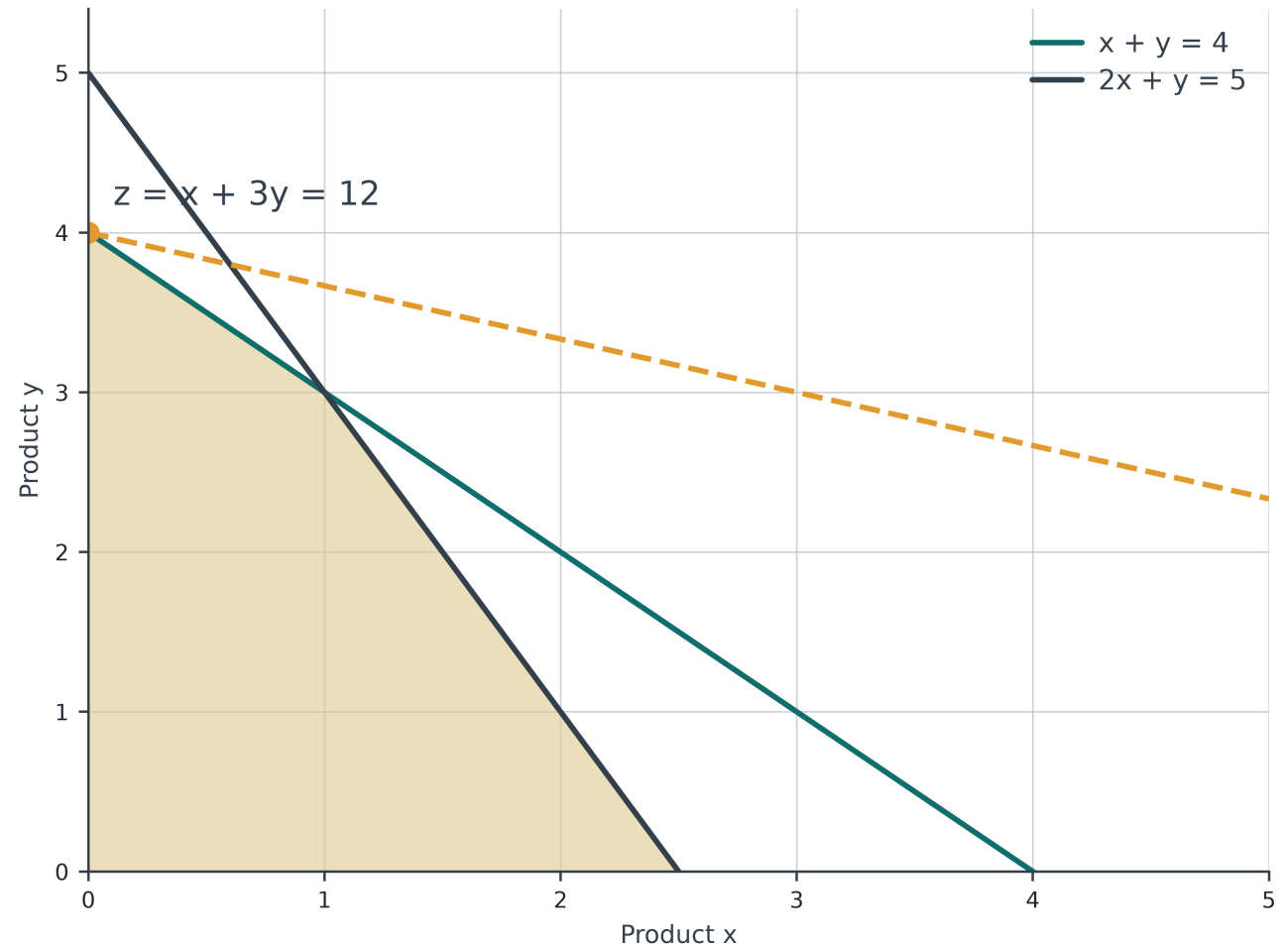
Practice solution: one pivot is enough

y enters; $\min(4/1, 5/1)=4$, so s_1 leaves.

$$y = 4 - x - s_1; \quad s_2 = 1 - x + s_1.$$

$$z = x + 3y = 12 - 2x - 3s_1 \leq 12.$$

Optimal: $(x, y) = (0, 4)$, slack $= (0, 1)$,
 $z^* = 12$.



Break

10 minutes

After the break: what is one more unit of a resource worth?

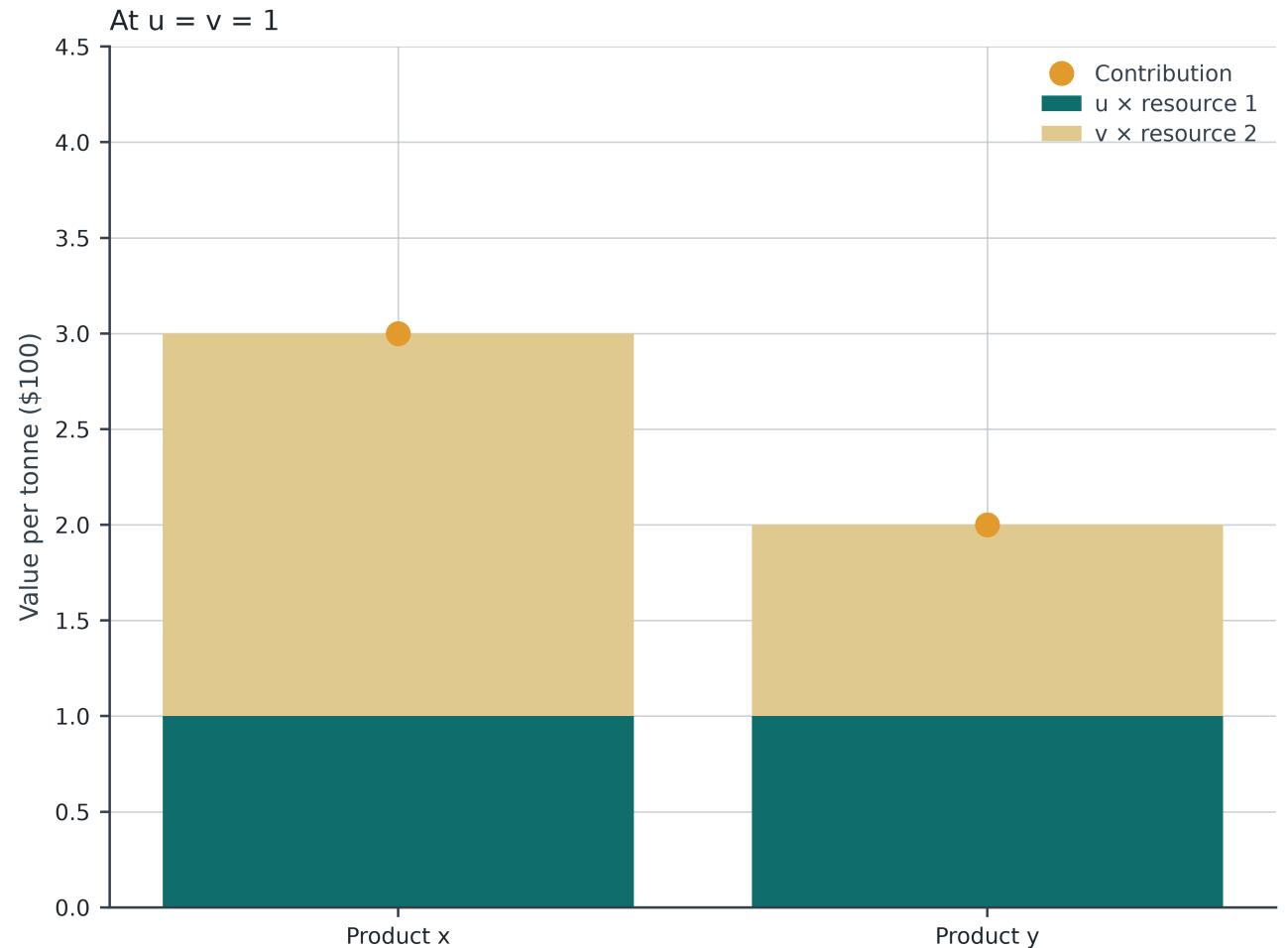
The dual prices the resources

u = price of resource 1; v = price of resource 2.

A unit of x uses resources priced at $u+2v$.

A unit of y uses resources priced at $u+v$.

Cover each contribution: $u+2v \geq 3$
and $u+v \geq 2$.



Primal and dual side by side

Primal: $\max c^T x$, subject to $Ax \leq b$, $x \geq 0$.

Dual: $\min b^T u$, subject to $A^T u \geq c$, $u \geq 0$.

Here: $\min w = 4u + 5v$

subject to $u + 2v \geq 3$, $u + v \geq 2$, $u, v \geq 0$.

Signs depend on the formulation

For a maximization primal:

Primal feature	Dual counterpart
Constraint $a_i^T x \leq b_i$	$u_i \geq 0$
Constraint $a_i^T x \geq b_i$	$u_i \leq 0$
Constraint $a_i^T x = b_i$	u_i is free
Variable $x_j \geq 0$	$(A^T u)_j \geq c_j$
Variable $x_j \leq 0$	$(A^T u)_j \leq c_j$
Variable x_j is free	$(A^T u)_j = c_j$

Weak duality gives an upper bound

For any primal-feasible x and dual-feasible u :

$$c^T x \leq x^T A^T u = u^T A x \leq u^T b.$$

Example: $(u,v)=(2,1)$ is feasible; $w=13$.

Therefore every feasible production plan has $z \leq 13$.

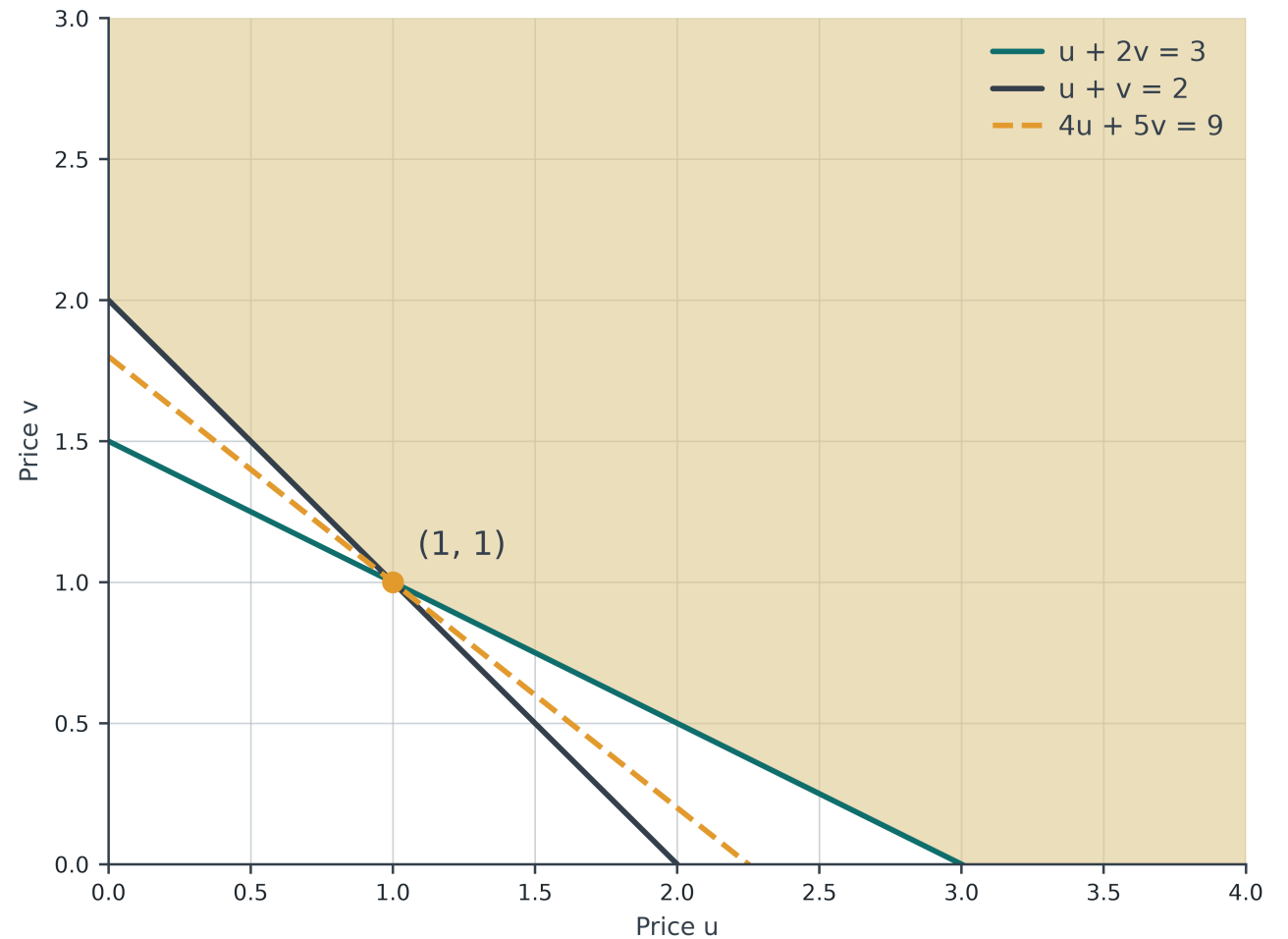
Dual geometry: tighten the upper bound

The dual feasible set lies above both price lines.

Minimize $4u+5v$ over that set.

Candidate corner prices: $(3,0)$, $(1,1)$, $(0,2)$.

Their bounds are 12, 9, and 10.



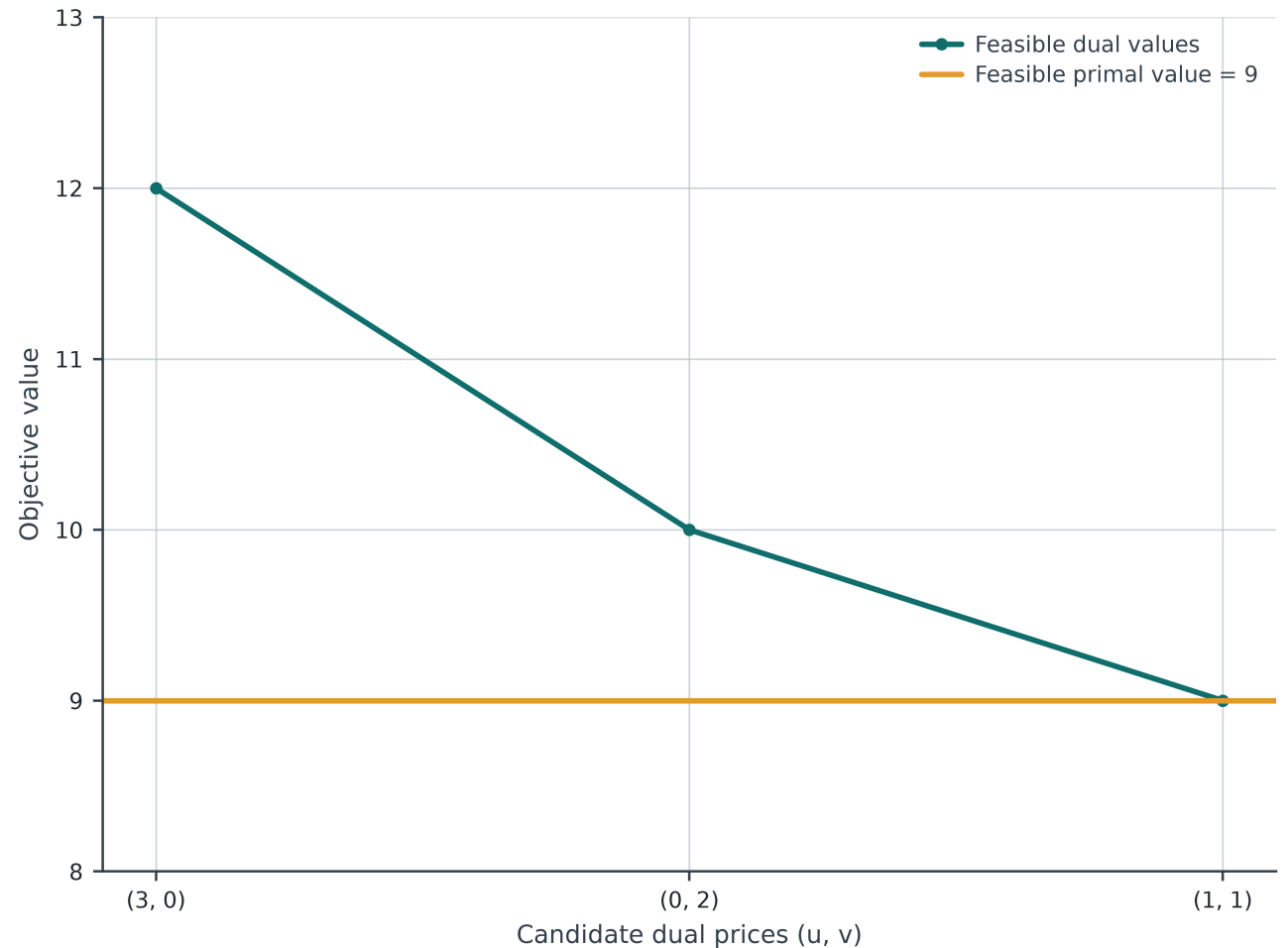
Solve the dual and compare

$$u+2v=3 \text{ and } u+v=2 \rightarrow u=v=1.$$

Dual value: $4(1)+5(1)=9$.

Primal point $(1,3)$ is feasible and gives 9.

Feasible values meet: both solutions are optimal.



Strong duality for linear programs

If an LP is feasible with a finite optimum, its dual also has an optimum with the same value.

A primal unbounded above implies the dual is infeasible.

Primal infeasible does not by itself imply dual unbounded.

The duality gap separates two kinds of waste

$$s = b - Ax \geq 0; \quad r = A^T u - c \geq 0.$$

$$w - z = u^T s + x^T r \geq 0.$$

Unused priced resources contribute $u^T s$.

Producing below the assigned resource cost contributes $x^T r$.

Complementary slackness

For primal-feasible x and dual-feasible u :

$u_i(b_i - a_i^T x) = 0$ for each resource.

$x_j((A^T u)_j - c_j) = 0$ for each product.

Unused resource implies zero price. Positive output implies a tight dual constraint.

A binding resource can have zero price.

LP duality is KKT in a special form

Rewrite $\max c^T x$ as $\min -c^T x$.

$$L = -c^T x + u^T (Ax - b) - r^T x.$$

Stationarity: $-c + A^T u - r = 0$.

Primal feasibility, $u, r \geq 0$, and complementarity complete the optimality conditions.

Practice: a dual certificate

For max $x+3y$ with the same two resource constraints:

1. Write the dual.
2. Test primal $(0,4)$ and dual $(u,v)=(3,0)$.
3. Check both objectives and all complementarity products.

Practice solution: zero price for unused capacity

Dual: $\min 4u+5v$; $u+2v \geq 1$, $u+v \geq 3$; $u,v \geq 0$.

At $(3,0)$: $w=12=z(0,4)$.

Resource slack: $(0,1)$. Dual slack: $(2,0)$.

Products: $3 \cdot 0=0$, $0 \cdot 1=0$, $0 \cdot 2=0$, $4 \cdot 0=0$.

Shadow prices: value per extra unit

Return to $c=(3,2)$, $b=(4,5)$: $u^*=v^*=1$.

Hold A and c fixed; keep the same basis feasible and optimal.

Then $\Delta z^* = u^*\Delta b_1 + v^*\Delta b_2$ exactly.

Each additional resource unit is worth \$100 within this region.

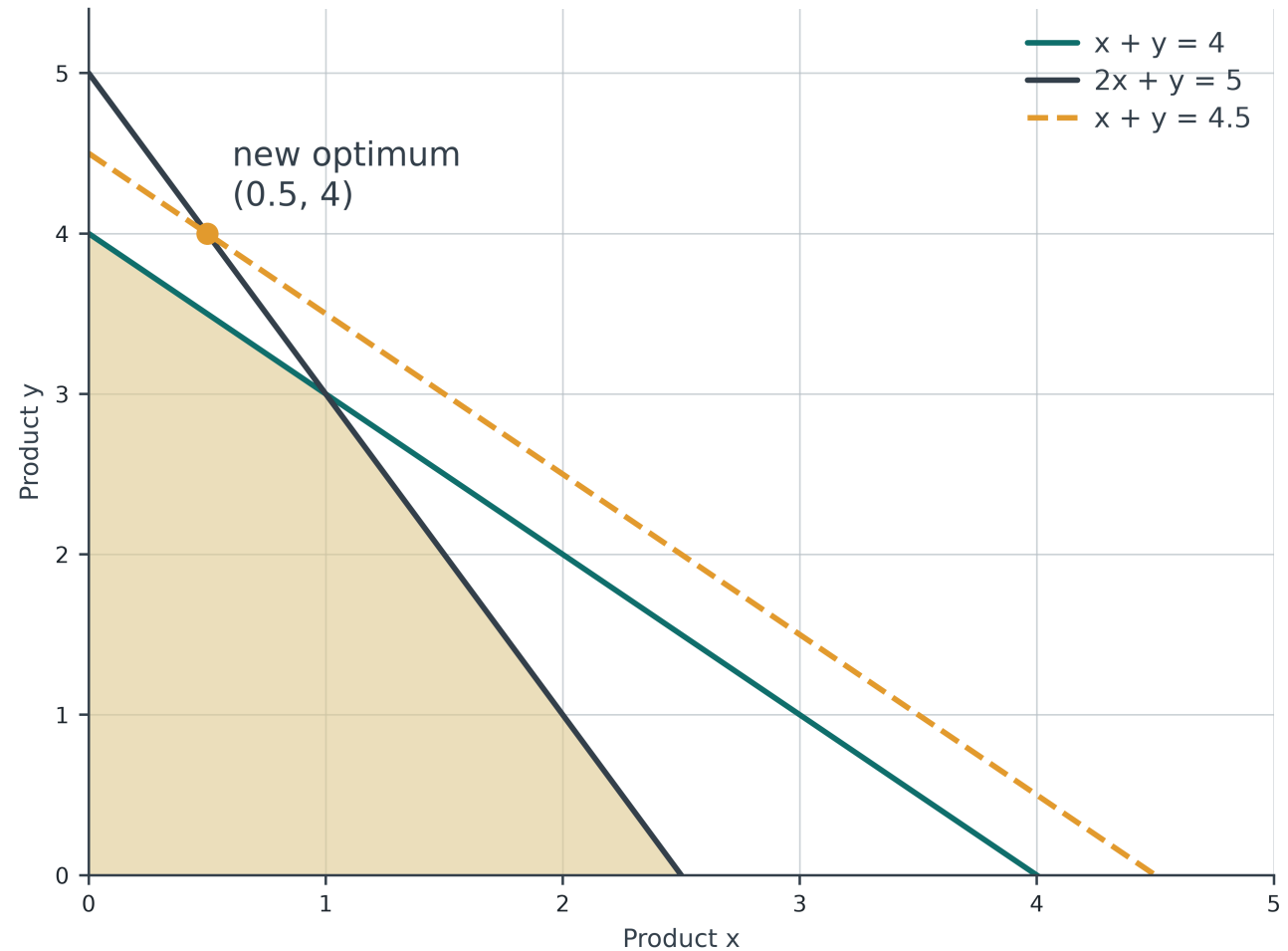
Allowable RHS range: resource 1

Hold $b_2=5$ and keep both resource constraints binding.

$x+y=b_1$; $2x+y=5 \rightarrow x=5-b_1$,
 $y=2b_1-5$.

$x,y \geq 0$ gives $2.5 \leq b_1 \leq 5$.

In this range: $z^*=b_1+5$. At $b_1=4.5$:
 $(x,y)=(0.5,4)$, $z^*=9.5$.



Beyond the range, the shadow price changes

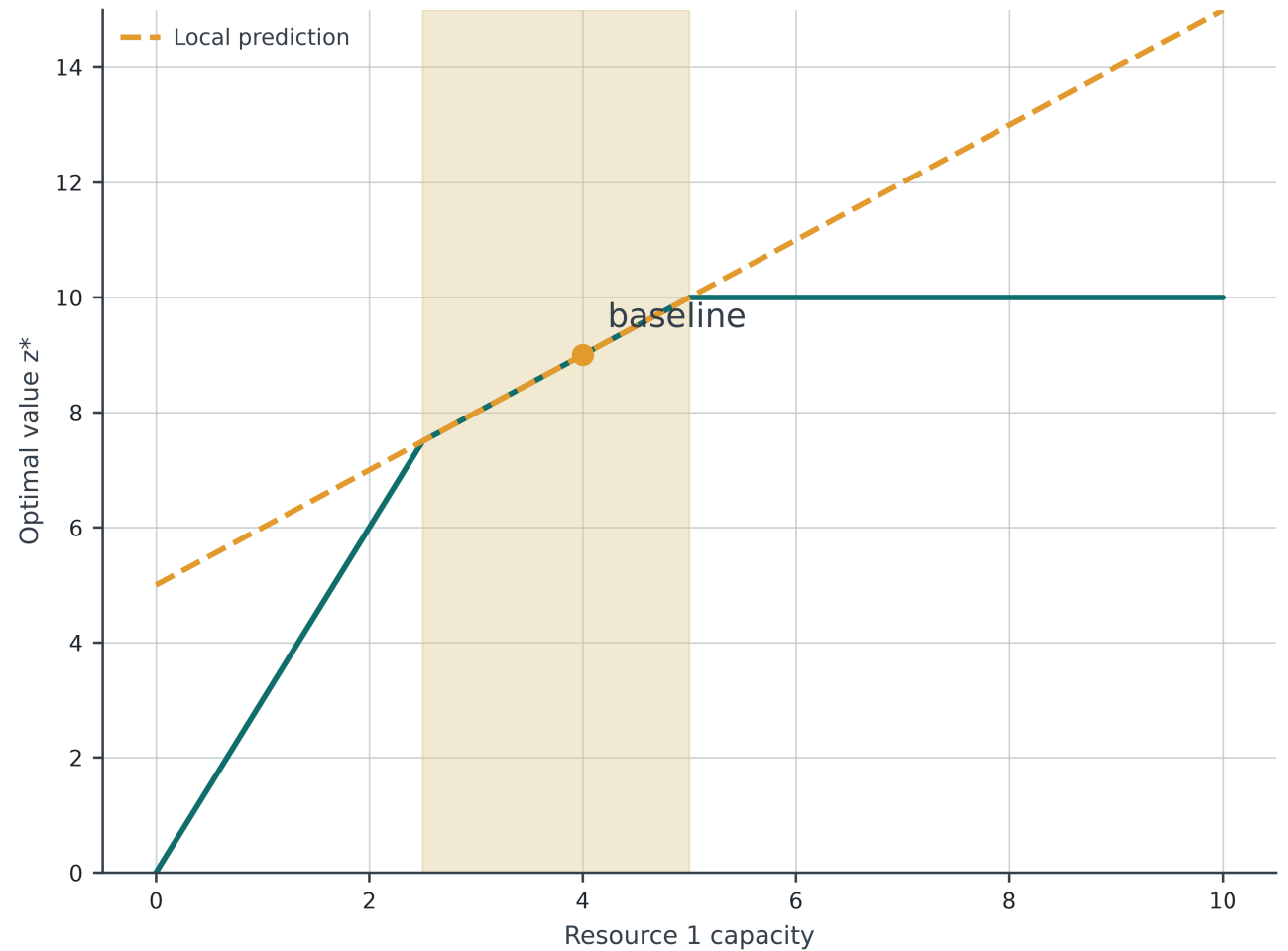
With $b_2=5$, hold A and c fixed.

$z^*(b_1)=3b_1$ for $0 \leq b_1 \leq 2.5$.

$z^*(b_1)=b_1+5$ for $2.5 \leq b_1 \leq 5$.

$z^*(b_1)=10$ for $b_1 \geq 5$.

At $b_1=5$: left slope=1, right slope=0.



Allowable RHS range: resource 2

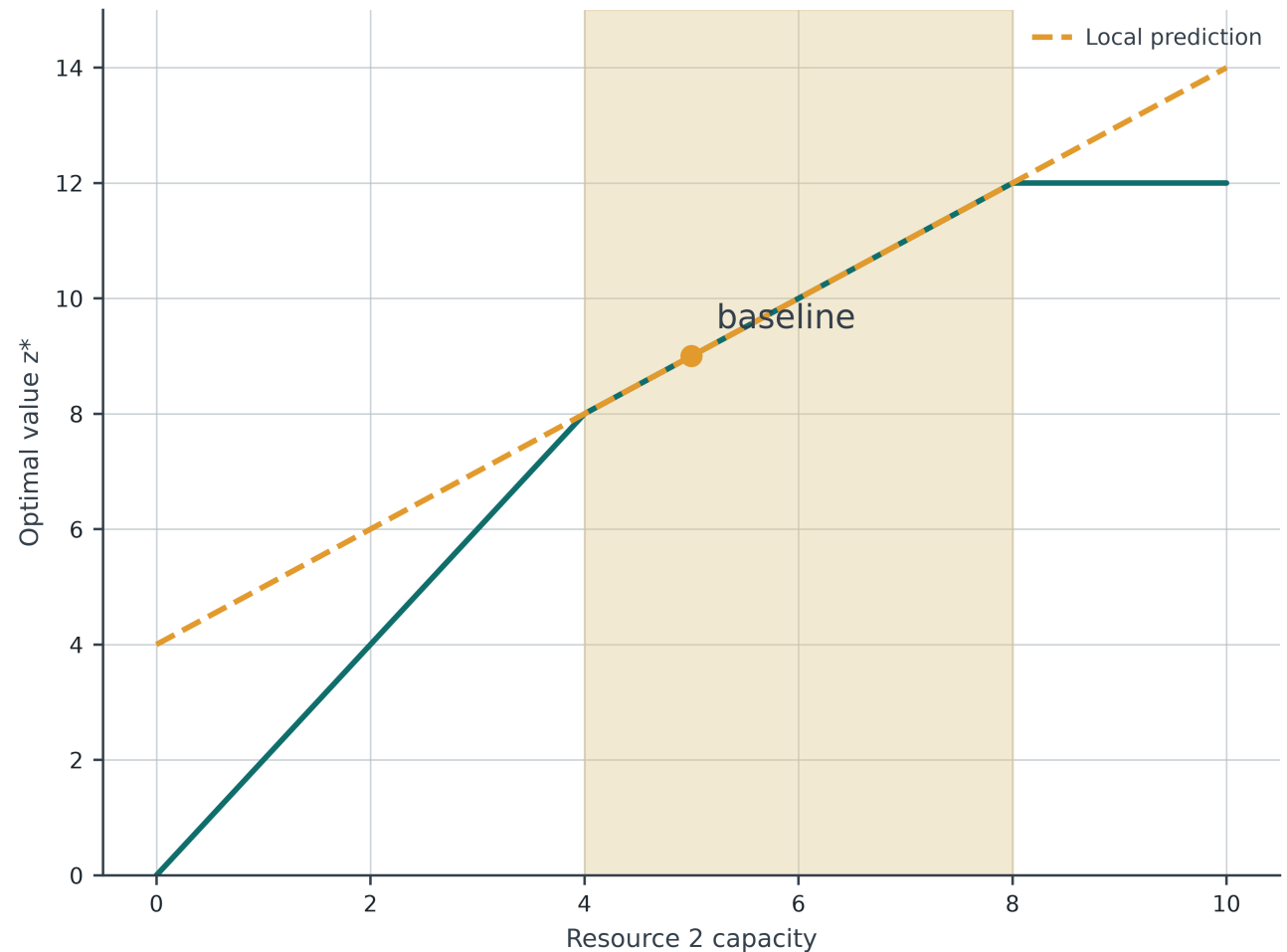
With $b_1=4$: $x=b_2-4$, $y=8-b_2$.

Same basis: $4 \leq b_2 \leq 8$; $z^*=b_2+4$.

At $b_2=6$: $(x,y)=(2,2)$, $z^*=10$.

If BOTH RHS values change: require $b_1 \leq b_2 \leq 2b_1$.

Separate allowable ranges cannot simply be combined.



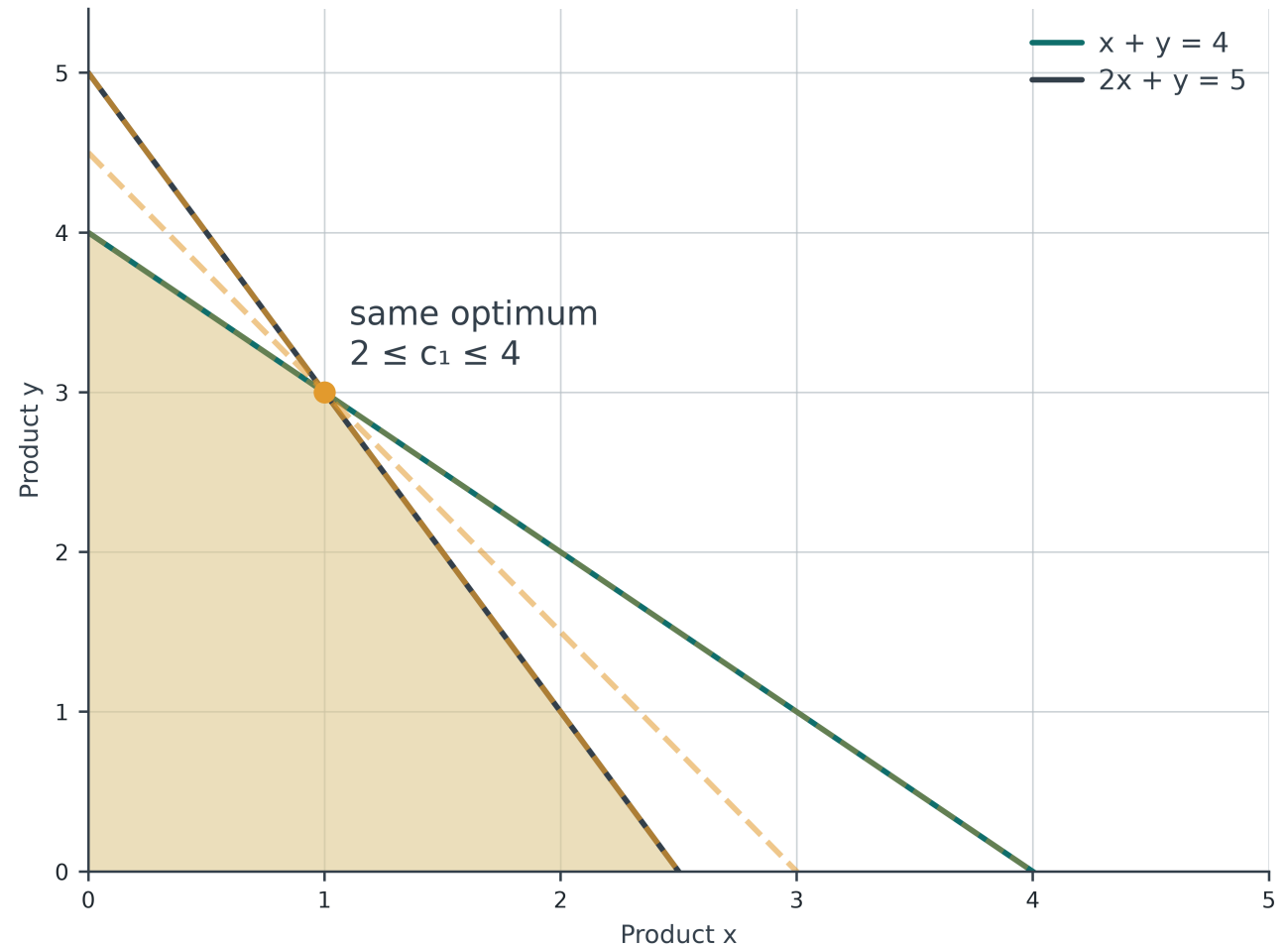
Objective sensitivity rotates the contour

Keep the candidate vertex (1,3).

$$u + 2v = c_1; \quad u + v = c_2 \rightarrow v = c_1 - c_2, \\ u = 2c_2 - c_1.$$

Dual feasibility requires $c_2 \leq c_1 \leq 2c_2$.

With $c_2=2$: $2 \leq c_1 \leq 4$. Endpoints give alternate optima.



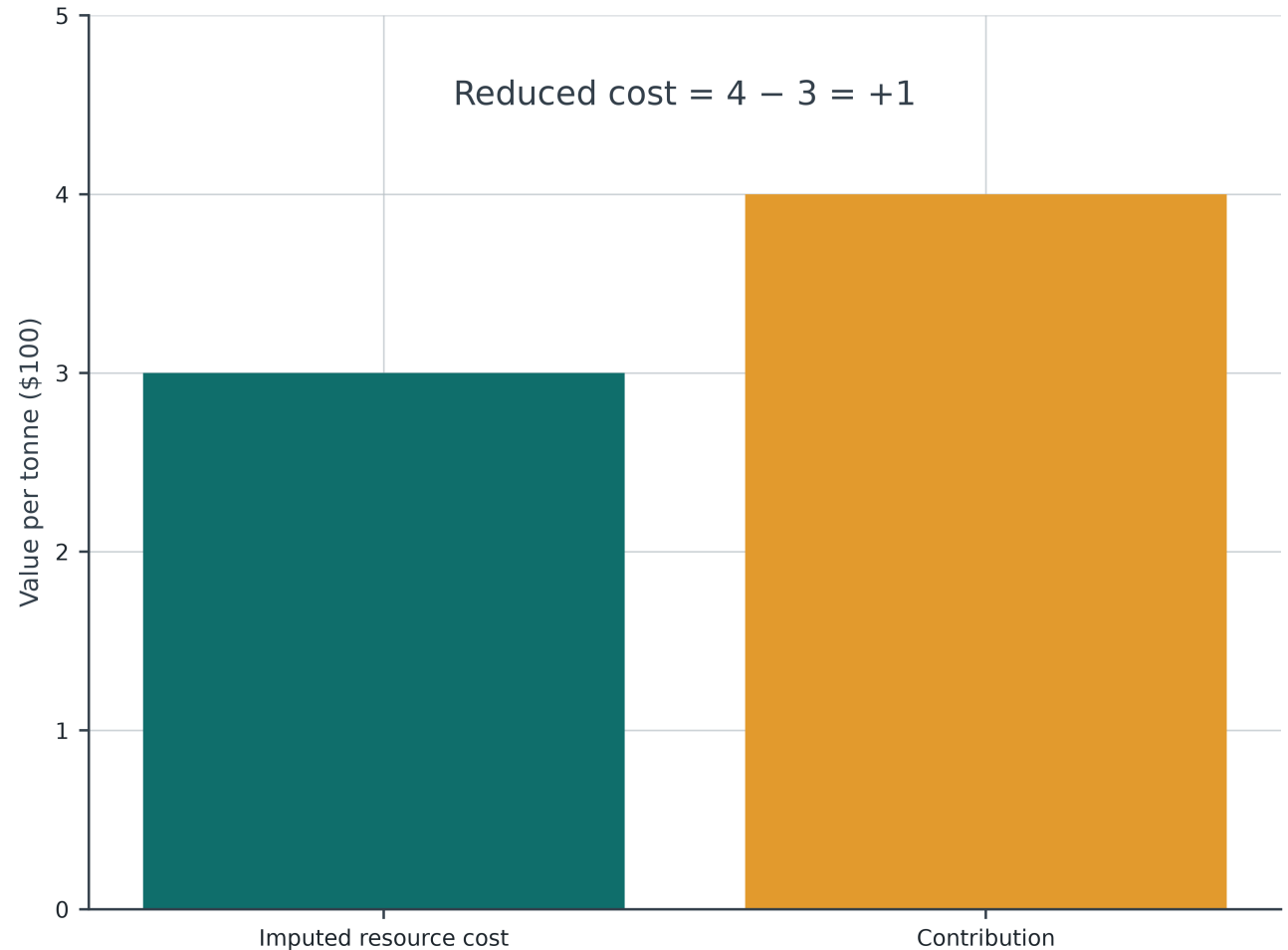
Reduced cost: should we introduce a new product?

New product t : $a_t=(1,2)$, $c_t=4$.

At current prices, resource cost= $1+2=3$.

Reduced cost= $4-3=+1$ for this max LP.

Here $x=1-t$, $y=3$, $0 \leq t \leq 1$ is feasible and $z=9+t$.



Read solver output with the sign convention

Output	Expected value	Meaning
success	True	Optimal termination
x	[1, 3]	Production
fun	−9	Minimized negative contribution
ineqlin.residual	[0, 0]	Unused capacity
ineqlin.marginals	[−1, −1]	Sensitivity of the minimized objective

SciPy minimizes $-3x-2y$. For max profit, shadow prices = $-\text{marginals} = (1,1)$.

Check status, feasibility, objective, and a small RHS change.

Practice: capacity purchase decision

Original LP, but an extra 0.5 unit/day of resource 1 costs \$30/day.

A. Predict production and gross contribution after purchase.

B. Calculate net benefit and decide whether to buy.

C. Would the same shadow price justify adding 2 units/day?

Capacity purchase: worked solution

A. $b_1=4.5 \rightarrow (x,y)=(0.5,4)$, $z^*=9.5=\$950/\text{day}$.

B. Gross gain=\$50/day; cost=\$30/day; net gain=\$20/day $\rightarrow$ buy.

C. $b_1=6 \rightarrow (x,y)=(0,5)$, $z^*=10=\$1,000/\text{day}$.

Gross gain=\$100/day. Buy only if the extra cost is below \$100/day.

Exit ticket: explain, calculate, certify

1. Why can an LP have an optimal edge?
2. For the original LP, test dual prices $(u,v)=(1,0)$.
3. Give a feasible primal-dual pair with equal value.
4. Predict z^* at $b_1=4.2$; explain why extrapolating to $b_1=6$ fails.

Exit ticket: answers and reasoning

1. A contour parallel to a feasible edge can be optimal along the whole edge.
2. $(1,0)$ fails $1+2\cdot 0 \geq 3$ and $1+0 \geq 2$; $w=4$ is not a valid bound.
3. Primal $(1,3)$, dual $(1,1)$: $z=w=9$.
4. $z^*(4.2)=9.2$; at $b_1=6$ the basis changes and $z^*=10$.

What an LP solution should tell us

A feasible production plan and its objective value.

A dual bound that meets that value.

Complementarity that explains active products and resources.

Sensitivity ranges that limit any resource-value claim.

References and optional follow-up

Boyd and Vandenberghe, Convex Optimization: Sections 4.3, 5.2, 5.5, and 5.6.

SciPy documentation: `linprog(method="highs")`, marginals and residuals.

Companion: `lp_duality_demo.py` reproduces and checks all numerical cases.