Optimality conditions

Lagrange multipliers and KKT

2105623 Optimization of Chemical Processes

19 October 2026 • 180-minute class

Learning outcomes

Find stationary points and classify them using Hessian eigenvalues.

Use Lagrange multipliers for equality constraints.

Apply all four KKT conditions with consistent signs.

Justify a minimum or maximum and interpret a multiplier.

Class schedule

0–40 min Optimality, Hessians, and eigenvalues

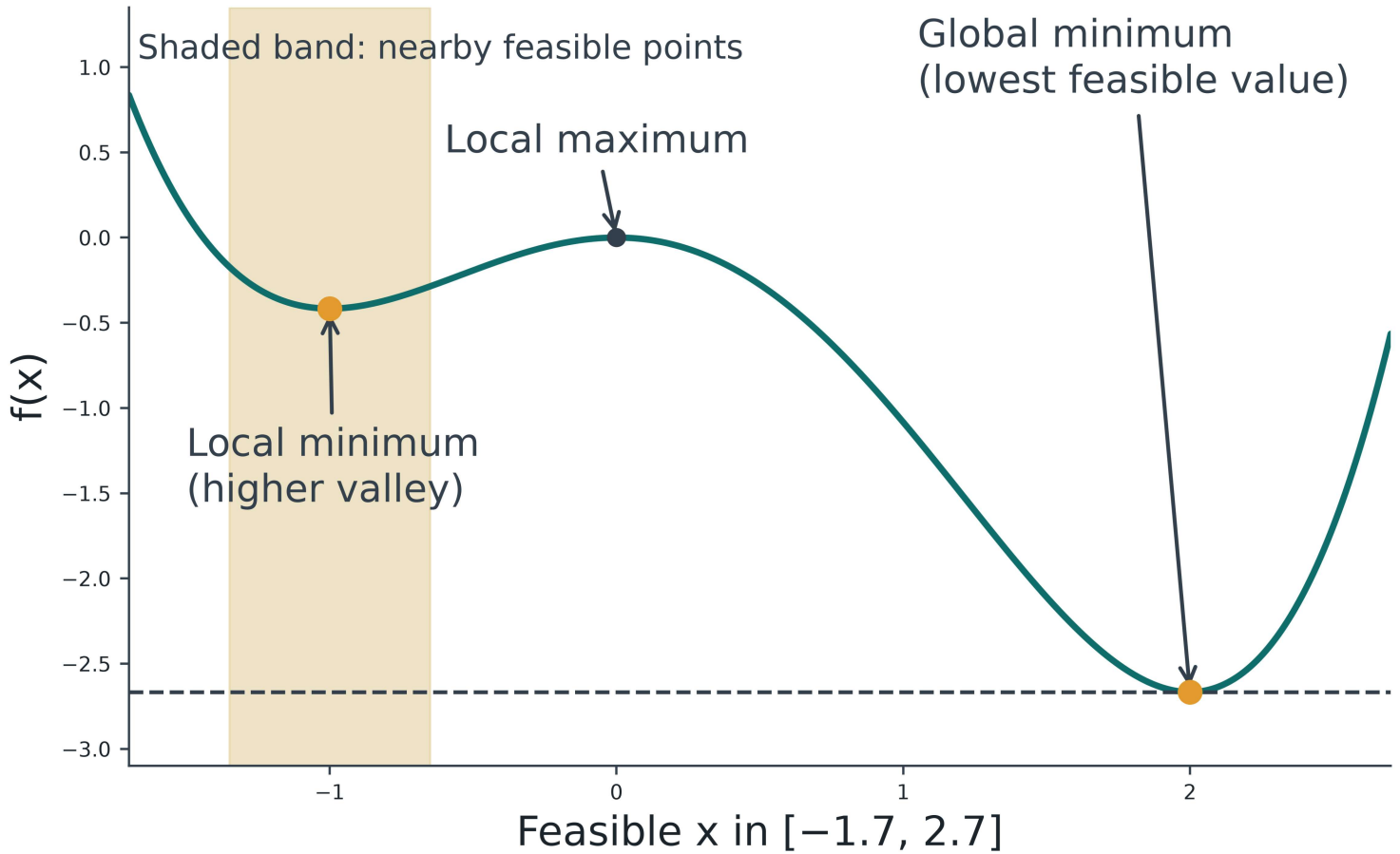
40–80 min Lagrange and production-load allocation

80–90 min Break

90–158 min KKT, delivery targets, and capacity value

158–180 min Practice, exit ticket, and feedback

Local and global optima



Local minimum

Lowest among nearby feasible points.

Global minimum

Lowest over the entire feasible interval.

For a maximum, reverse the comparison.

Every global minimum is local. A local minimum need not be global.

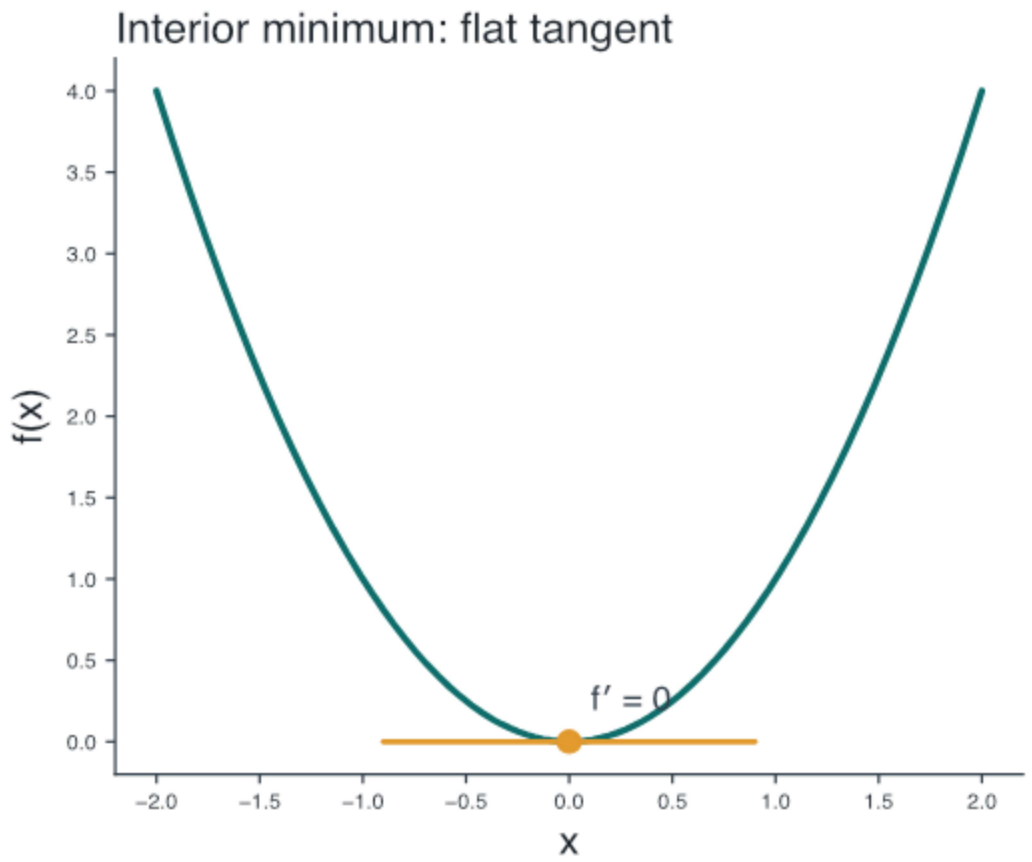
First-order condition

At a differentiable interior local optimum:

$\nabla f(x^*) = 0$

In one variable: $f'(x^*) = 0$

A feasible boundary optimum can have $\nabla f(x^*) \neq 0$.



A nonzero slope gives a local direction of improvement.

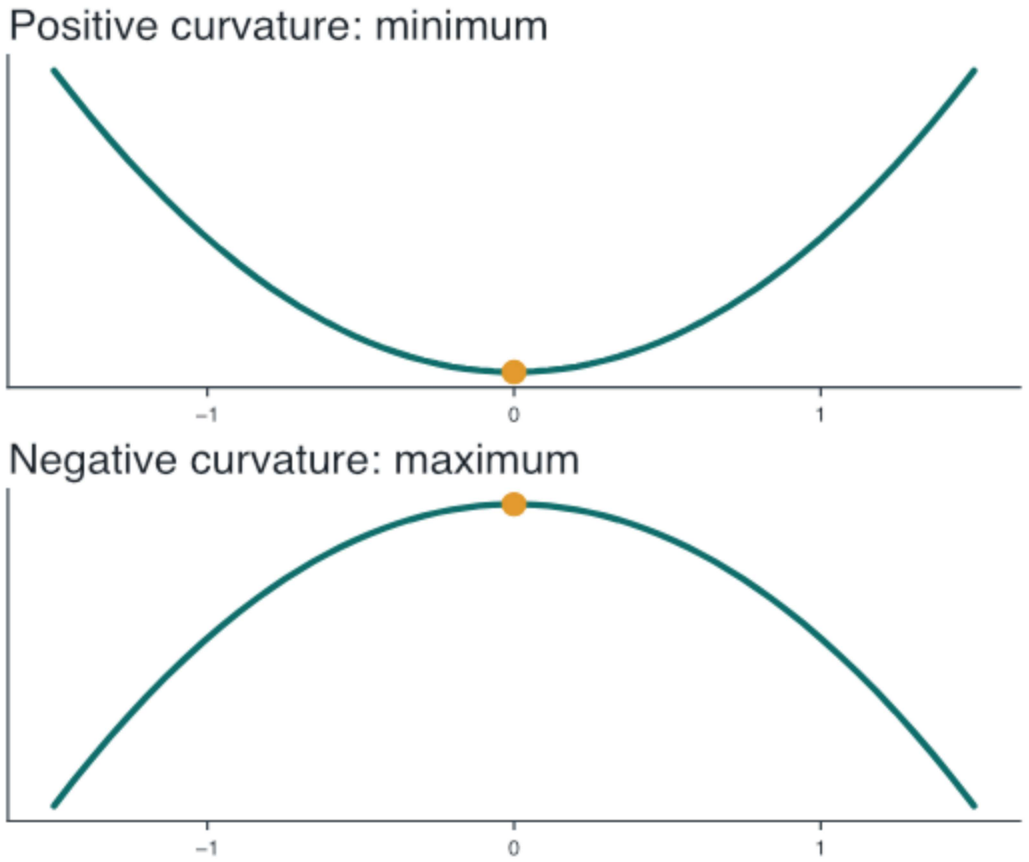
Second-order conditions

Assume f is twice continuously differentiable near x*.

Necessary at an interior minimum: $\nabla f = 0$ and $H \succeq 0$.

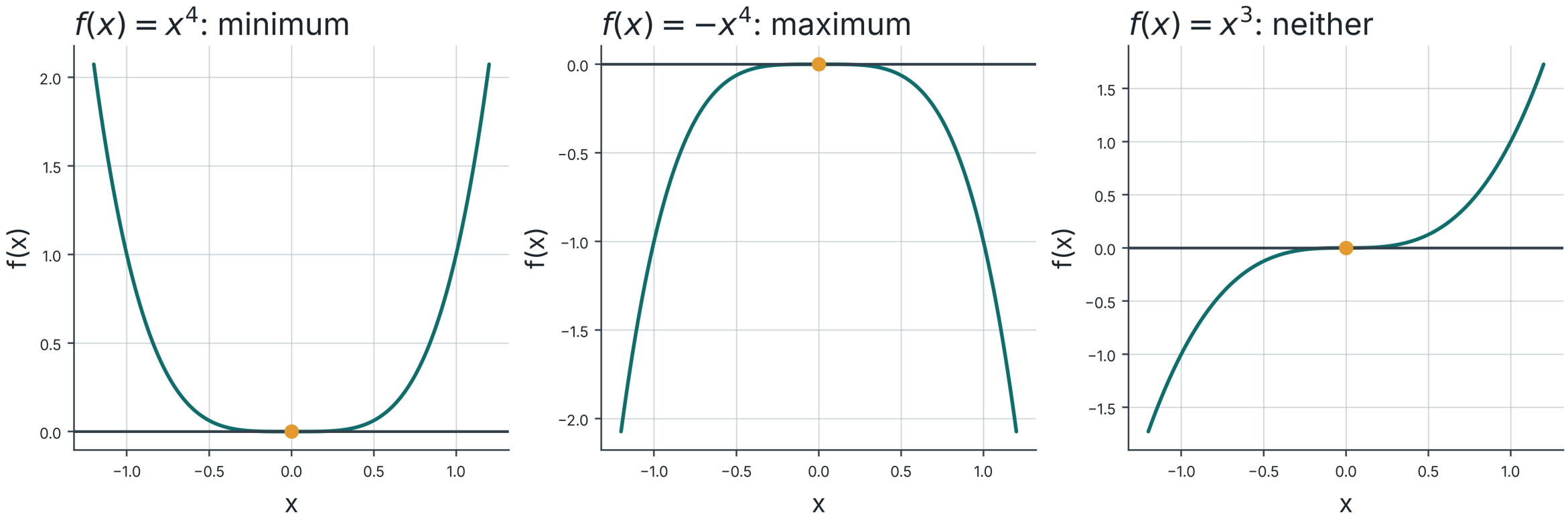
Sufficient for a strict local minimum: $\nabla f = 0$ and $H \succ 0$.

For a maximum, use $H \preceq 0$ and $H \prec 0$.



At a stationary point, the sign of curvature classifies strict extrema.

When the second derivative is zero

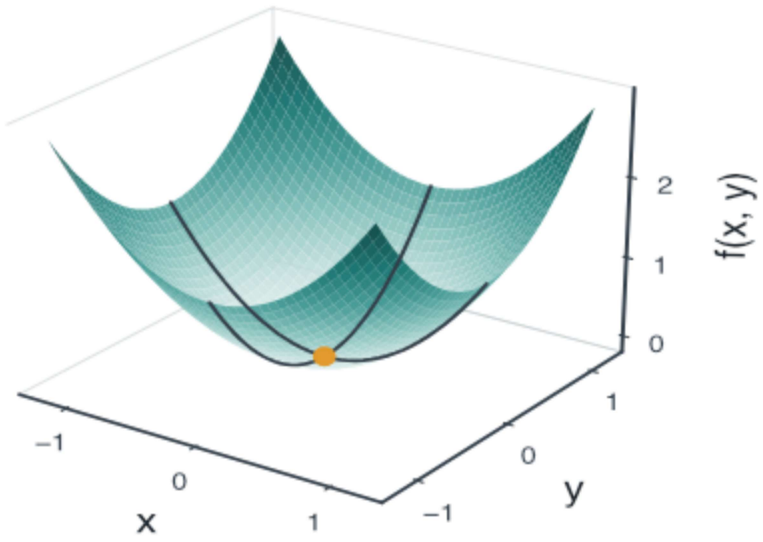


Two-variable Hessian test

At a stationary point: $\nabla f = 0$. $H = [[f_{xx}, f_{xy}], [f_{yx}, f_{yy}]]$, $D = f_{xx}\, f_{yy} - (f_{xy})^2$

Strict local minimum

$f(x, y) = x^2 + y^2$

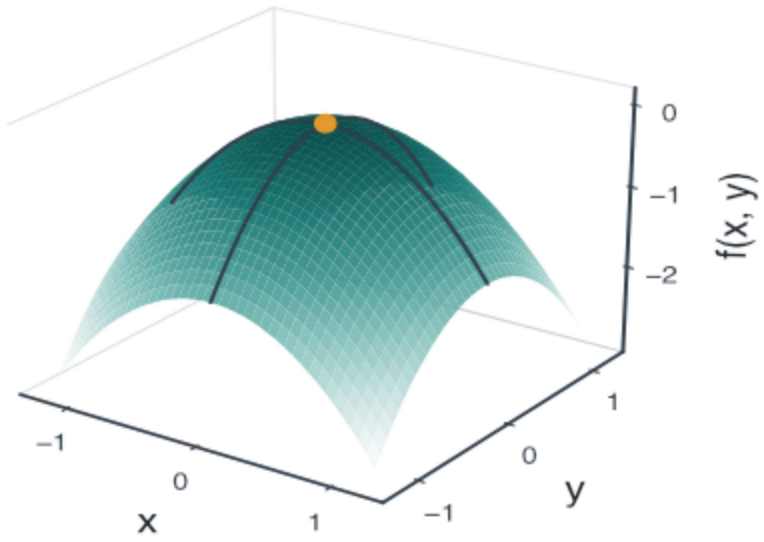


$H = \mathrm{diag}(2, 2)$

Positive definite: $D > 0$, $f_{xx} > 0$

Strict local maximum

$f(x, y) = -x^2 - y^2$

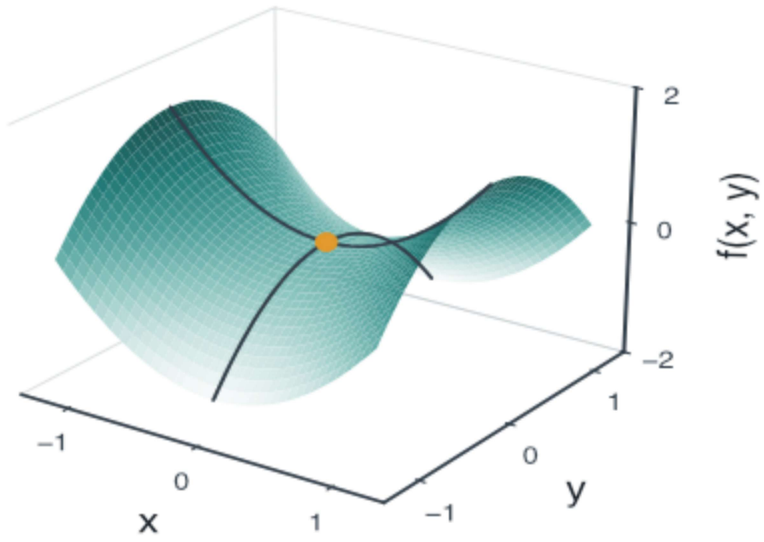


$H = \mathrm{diag}(-2, -2)$

Negative definite: $D > 0$, $f_{xx} < 0$

Saddle point

$f(x, y) = x^2 - y^2$



$H = \mathrm{diag}(2, -2)$

Indefinite: $D < 0$

$D = 0$: the Hessian test is inconclusive. Amber dots mark (0, 0, 0).

Eigenvalues: meaning and calculation

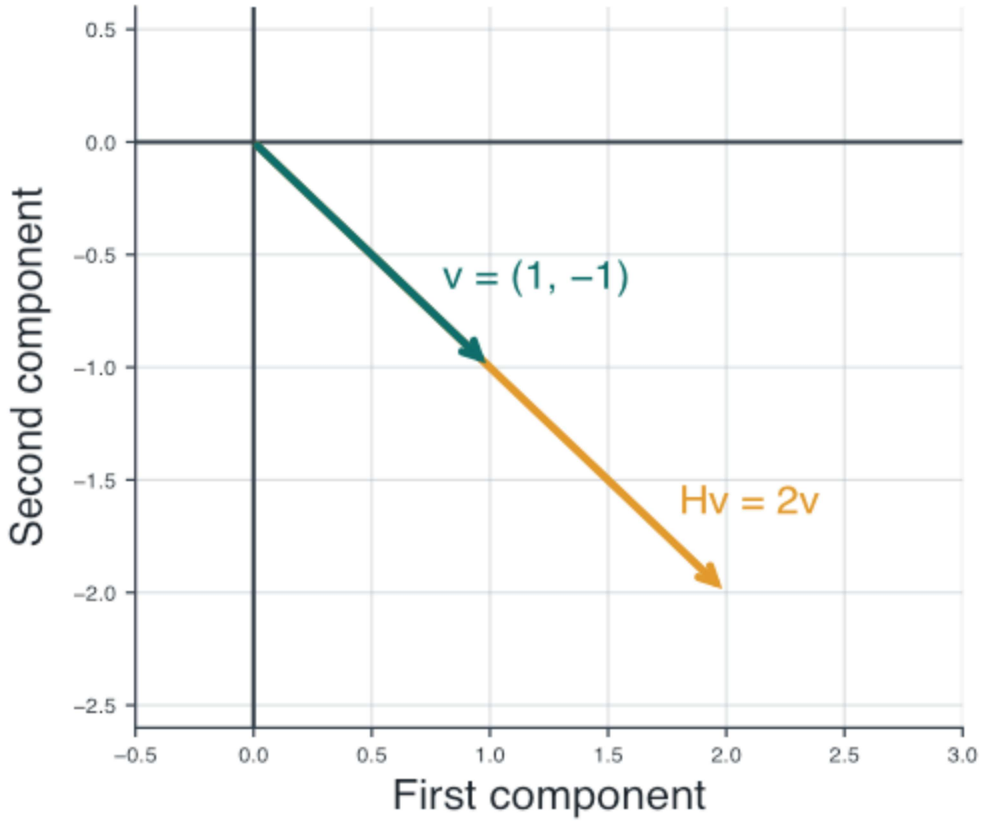
$Hv = \lambda v$, with $v \neq 0$.

v is an eigenvector: H scales its direction by λ.

$(H - \lambda I)v = 0$ needs a nonzero solution.

Therefore $\det(H - \lambda I) = 0$.

For a C^2 function, H is symmetric and its eigenvalues are real.



Example: H = [[4, 2], [2, 4]] and λ = 2.

Computing eigenvalues of a 2 × 2 Hessian

$H = [[a, b], [b, c]]$

$\det(H - \lambda I) = (a-\lambda)(c-\lambda) - b^2$

$\lambda^2 - (a+c)\lambda + (ac-b^2) = 0$

$\lambda_{1,2} = [(a+c) \pm \sqrt{((a-c)^2 + 4b^2)}] / 2$

Check: $\lambda_1 + \lambda_2 = \mathrm{tr}(H)$; $\lambda_1\lambda_2 = \det(H)$.

Calculation sequence

1. Evaluate H at the stationary point.
2. Subtract λ from diagonal entries.
3. Set the determinant to zero.
4. Solve and check both roots.

Worked example: two positive eigenvalues

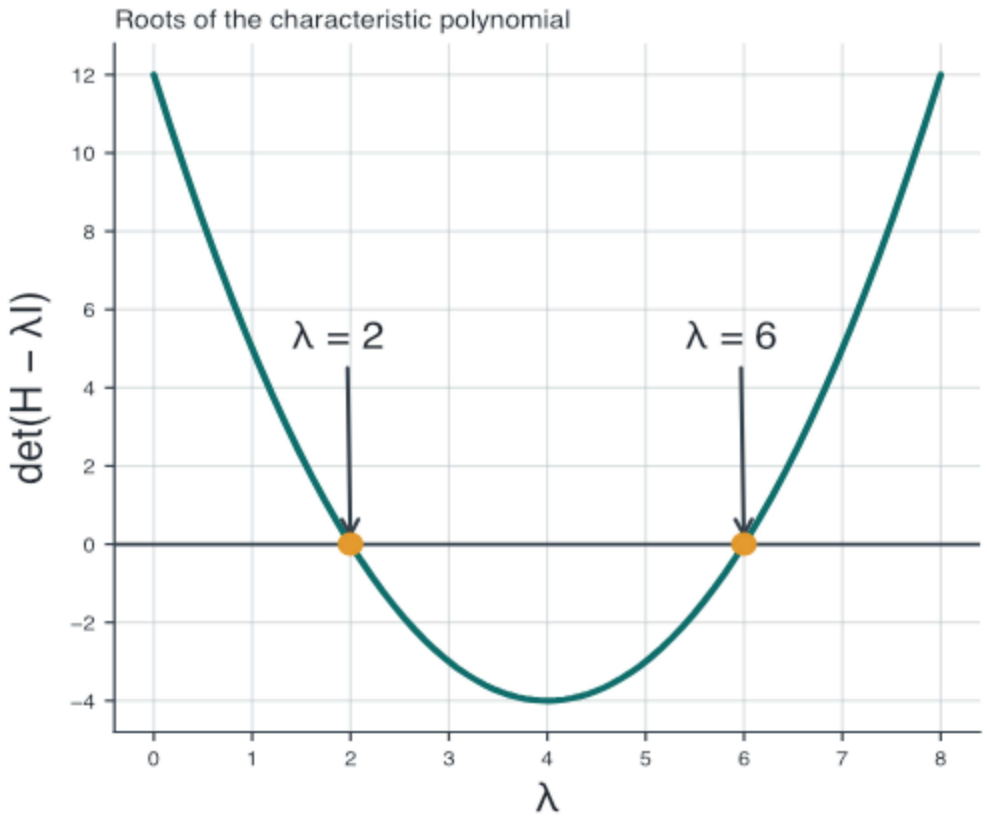
$f(x,y) = 2x^2 + 2xy + 2y^2$; $\nabla f(0,0) = 0$.

$H = [[4, 2], [2, 4]]$

$\det(H - \lambda I) = (4-\lambda)^2 - 4$

$\lambda^2 - 8\lambda + 12 = (\lambda-2)(\lambda-6) = 0$

$\lambda_1 = 2$, $\lambda_2 = 6 > 0$: strict local minimum.

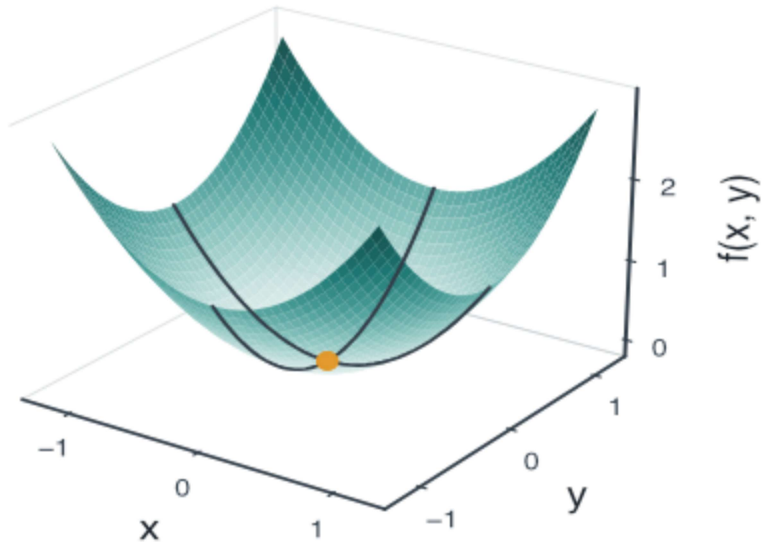


Both roots are positive. This quadratic also has a unique global minimum.

Eigenvalue signs and local curvature

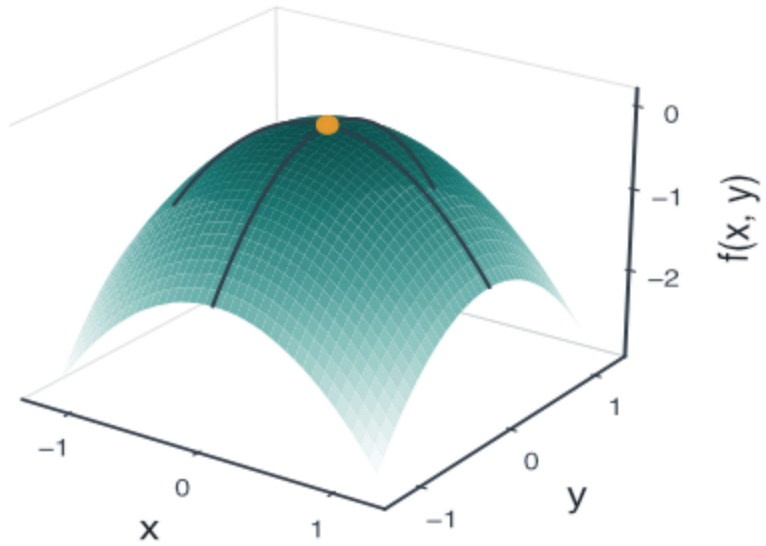
At an interior stationary point: $d^T H d = \Sigma \lambda_i (q_i^T d)^2$, with orthonormal eigenvectors q_i.

All $\lambda_i > 0$



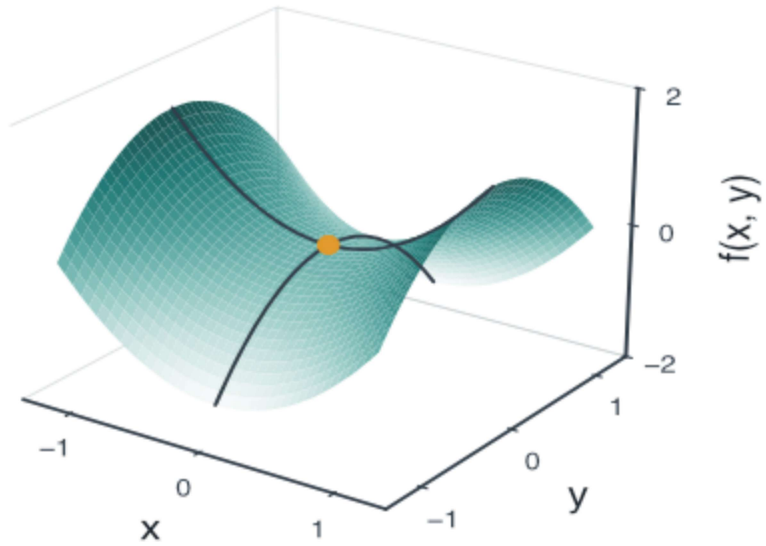
Strict local minimum

All $\lambda_i < 0$



Strict local maximum

Positive and negative λ_i



Saddle point

Zero eigenvalues without mixed signs: the second-order test is inconclusive.

Practice: calculate and classify

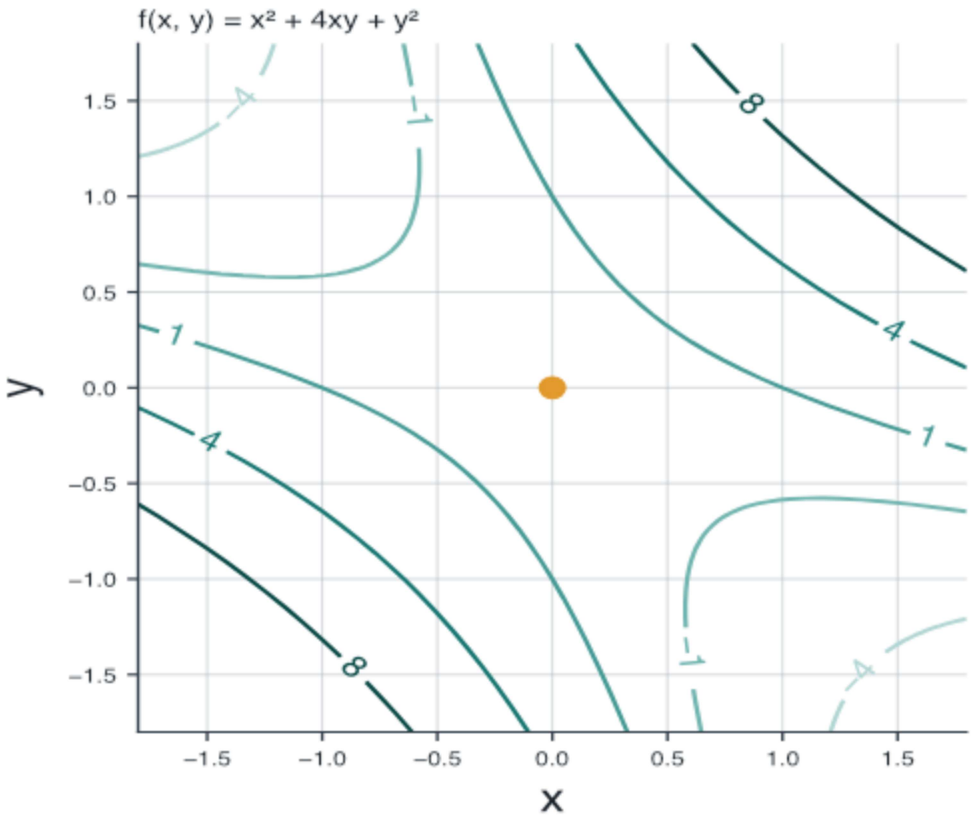
$f(x,y) = x^2 + 4xy + y^2$

1. Verify that (0,0) is stationary.

2. Compute the Hessian H.

3. Find $\det(H - \lambda I)$, then both eigenvalues.

4. Classify (0,0) and check along $y = x$ and $y = -x$.



Predict the classification, then verify it with eigenvalues.

Practice solution: a saddle point

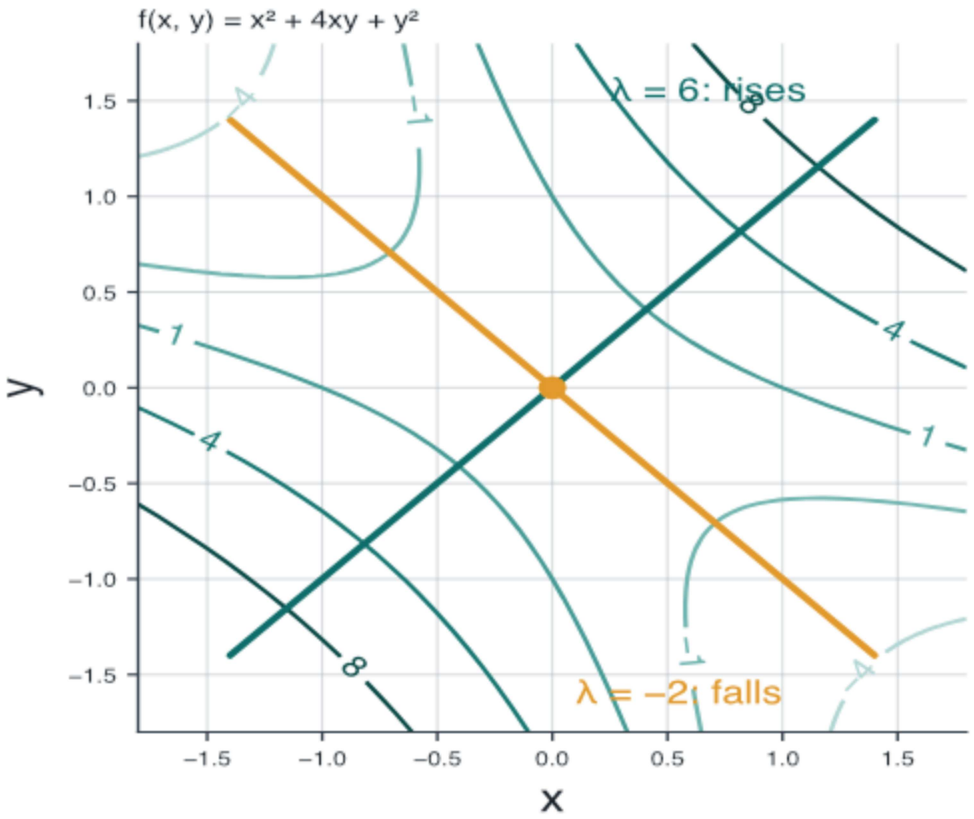
$\nabla f = (2x+4y,\ 4x+2y)$; $\nabla f(0,0) = 0$.

$H = [[2, 4], [4, 2]]$

$\det(H - \lambda I) = (2-\lambda)^2 - 16$
$= \lambda^2 - 4\lambda - 12 = (\lambda-6)(\lambda+2)$

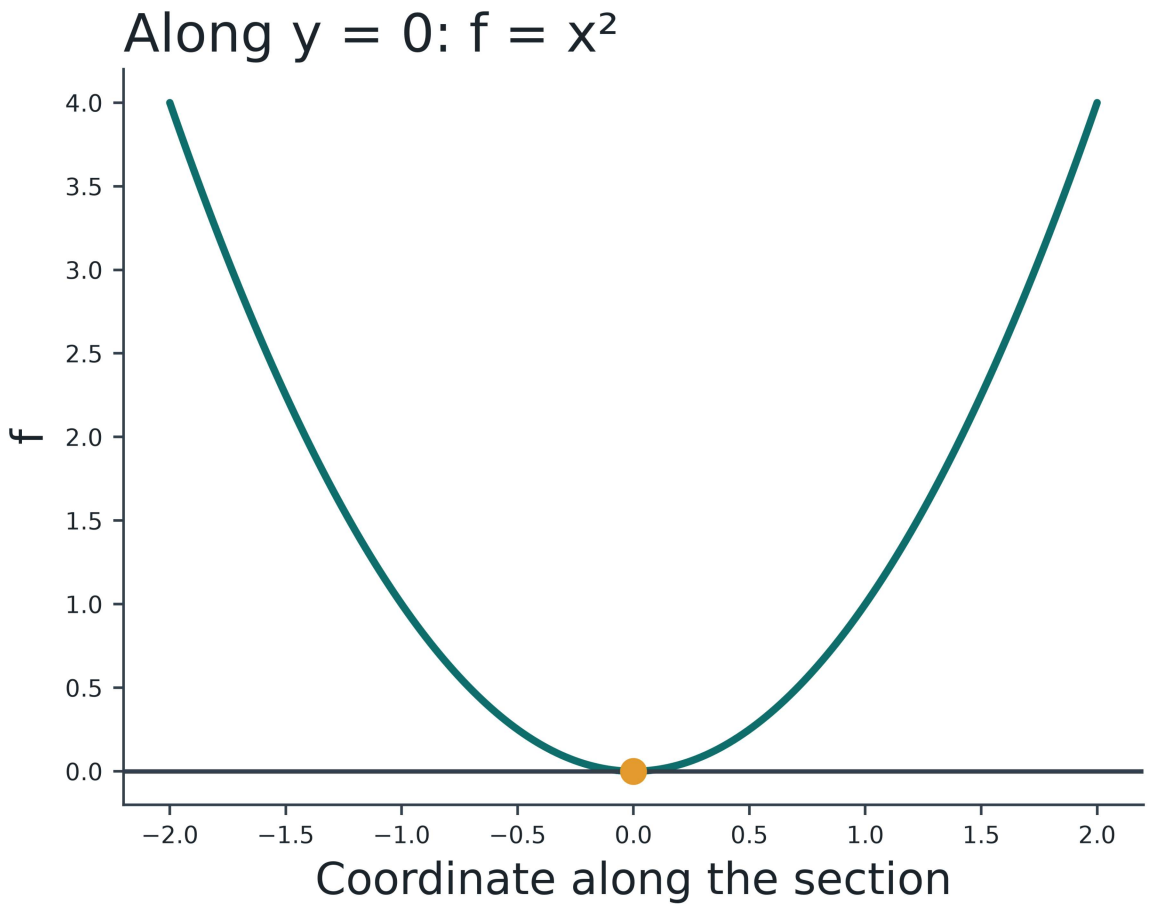
$\lambda_1 = 6$, $\lambda_2 = -2$: mixed signs, so a saddle.

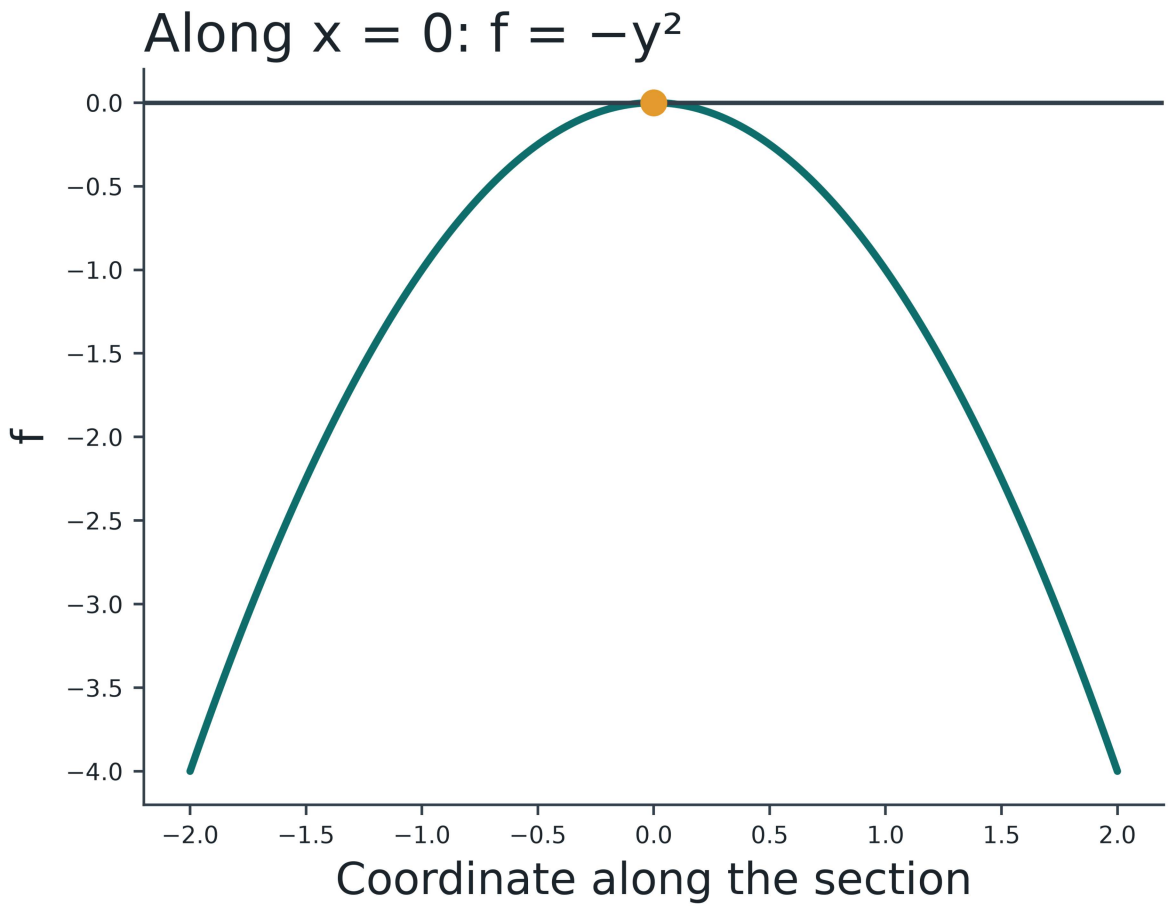
Along $y = x$: $f = 6x^2 > 0$.
Along $y = -x$: $f = -2x^2 < 0$ $(x \neq 0)$.



Two eigenvector directions reveal opposite curvature.

A saddle falls in one direction and rises in another





Worked example: quadratic cost

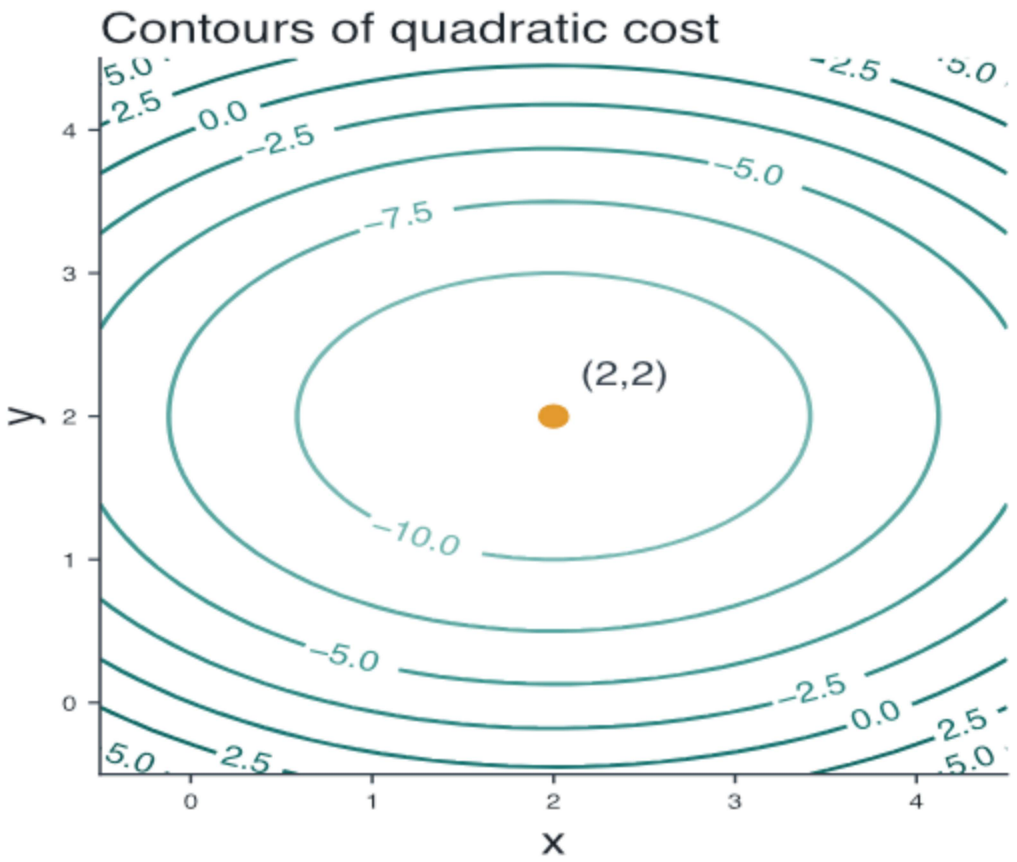
Minimize $f(x,y) = x^2 + 2y^2 - 4x - 8y$.

$\nabla f = (2x - 4, 4y - 8) = (0,0) \rightarrow (2,2)$

$H = \mathrm{diag}(2,4) \succ 0$

$f = (x - 2)^2 + 2(y - 2)^2 - 12$

Unique global minimum: $f(2,2) = -12$. No maximum.



All contours surround the unique global minimum.

Global extrema on a closed interval

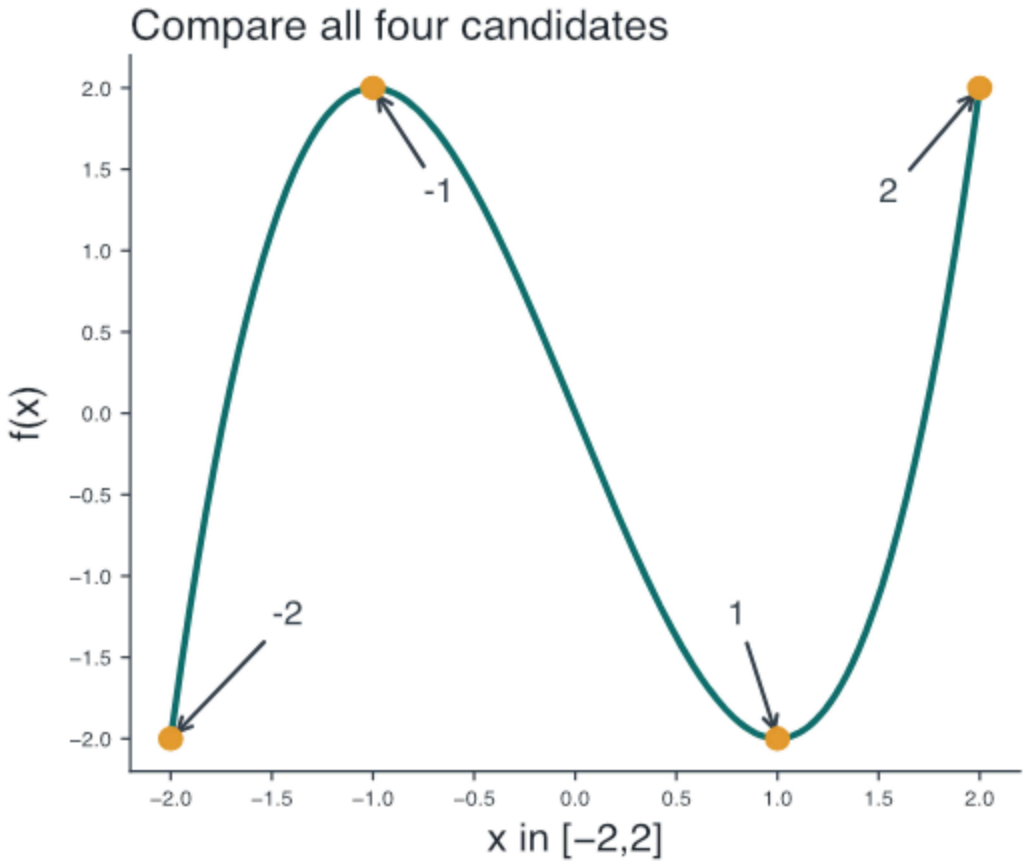
Find max and min of $f(x) = x^3 - 3x$ on $[-2,2]$.

Interior candidates: $f' = 3x^2 - 3 = 0 \rightarrow x = -1, 1$

Include both endpoints: $x = -2, 2$.

Values at $x = -2, -1, 1, 2$ are $-2, 2, -2, 2$.

Global min = -2 at -2 and 1. Global max = 2 at -1 and 2.



Endpoints must be compared with interior stationary points.

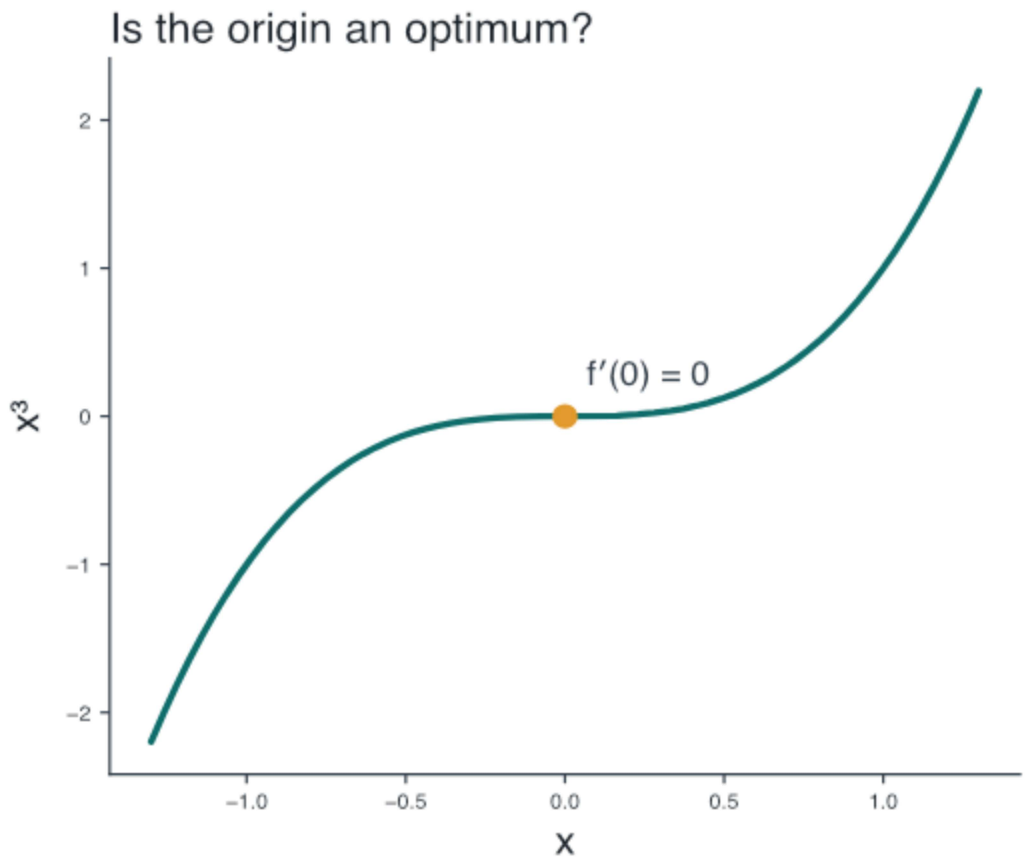
Checkpoint: identify the missing argument

Why does ∇f = 0 alone not prove a minimum?

What does a zero Hessian determinant tell us?

Why must endpoints be checked on [a,b]?

What extra structure turns a local minimum into a global one?



A flat tangent alone does not prove a minimum or maximum.

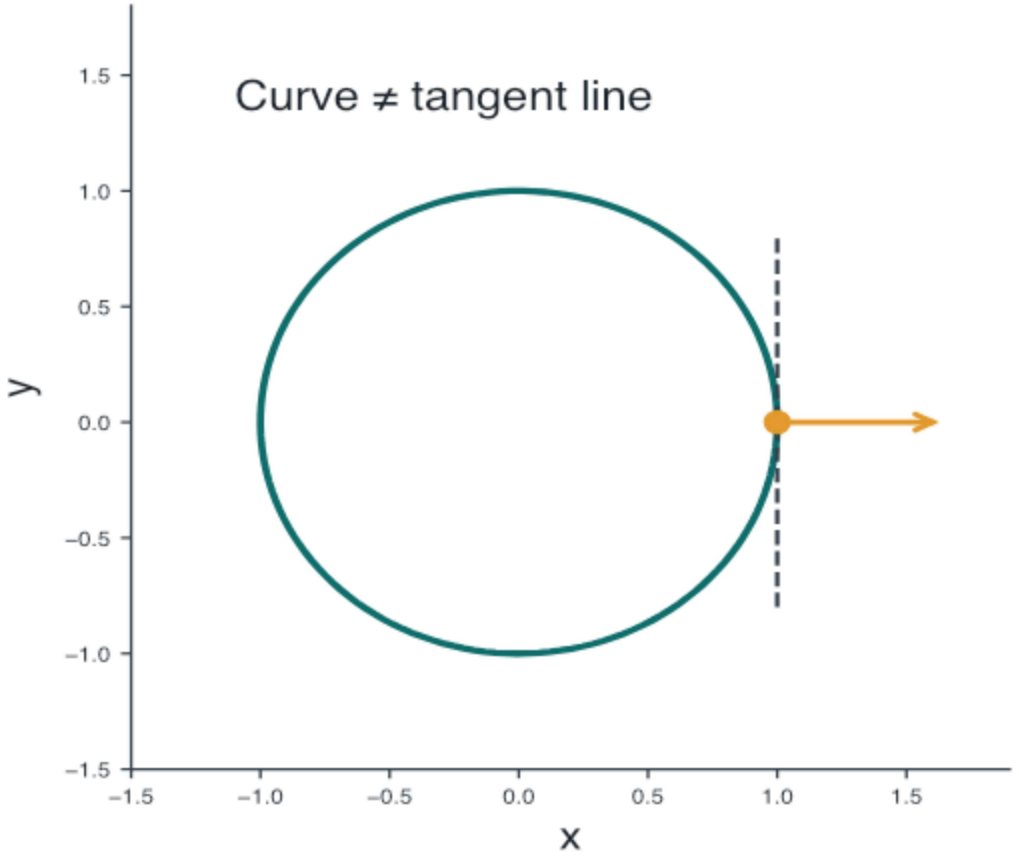
Equality-constrained optimization

Minimize f(x) subject to h(x) = 0.

A tangent direction d satisfies $\nabla h(x^*)^T d = 0$.

At a regular optimum, $\nabla f(x^*)$ is normal to the surface.

Write $\nabla f(x^*) + \lambda \nabla h(x^*) = 0$.



Feasible motion follows the curve; its instantaneous direction is tangent.

The Lagrangian

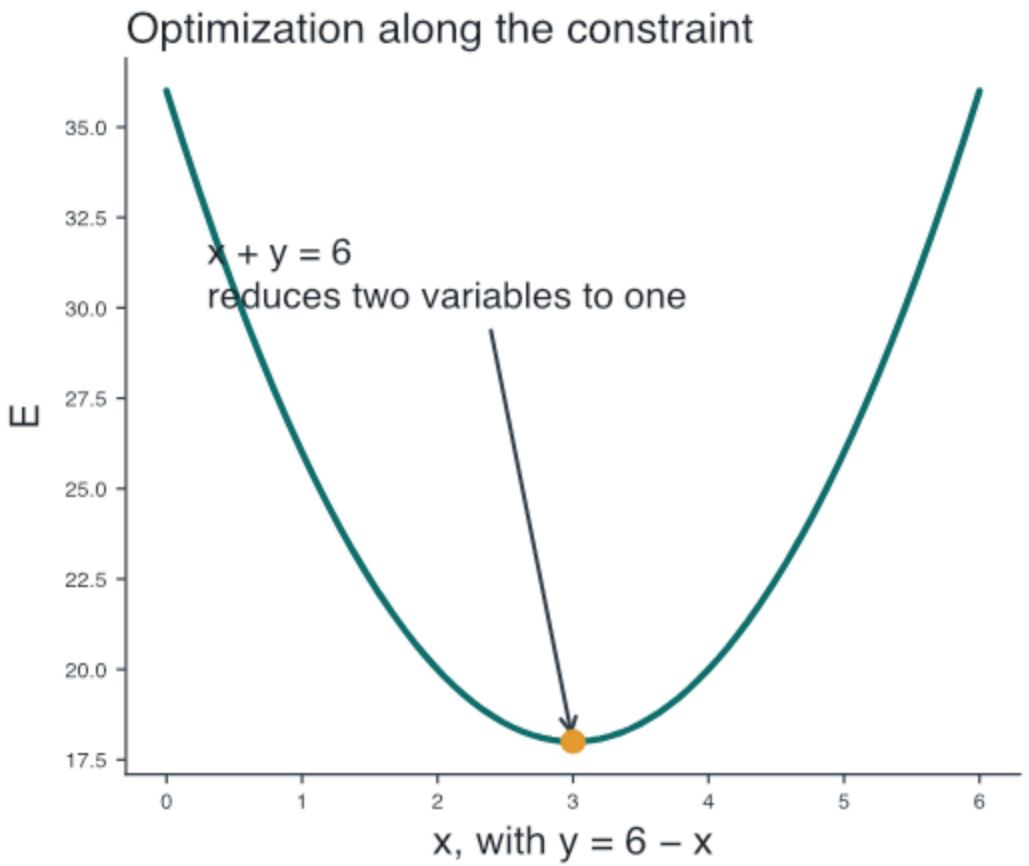
$L(x,\lambda) = f(x) + \lambda h(x)$

Solve $\nabla_x L(x,\lambda) = 0$ and $h(x) = 0$.

Equality multipliers λ can be positive, zero, or negative.

For several equalities, use $L = f + \Sigma \lambda_j h_j$.

Necessity assumes independent equality gradients at x^*.



The multiplier enforces stationarity along feasible motion.

Lagrange multipliers: geometric meaning

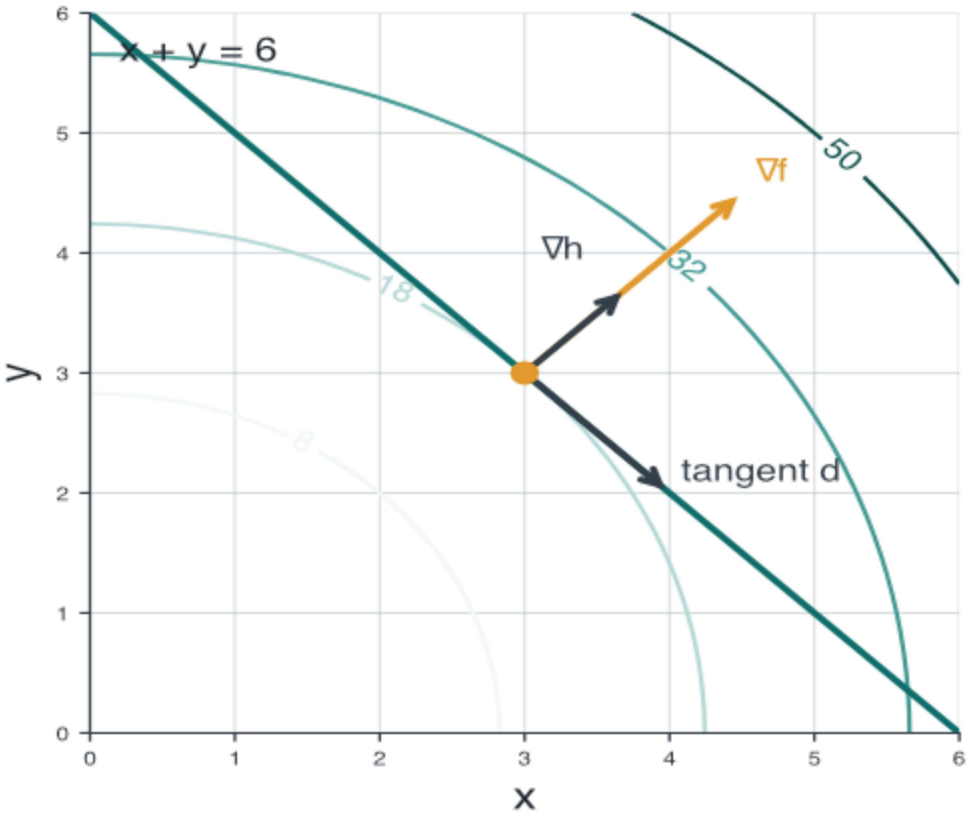
Minimize $f = x^2+y^2$ with $h = x+y-6 = 0$.

At a constrained optimum, feasible motion cannot lower f to first order.

A tangent direction d satisfies $\nabla h^T d = 0$.
At the optimum, $\nabla f^T d = 0$ as well.

Thus ∇f is parallel to ∇h:
$\nabla f + \lambda \nabla h = 0$.

At (3,3): (6,6) + (−6)(1,1) = (0,0).



The level curve f = 18 touches x+y = 6. Arrows illustrate direction.

The multiplier measures sensitivity

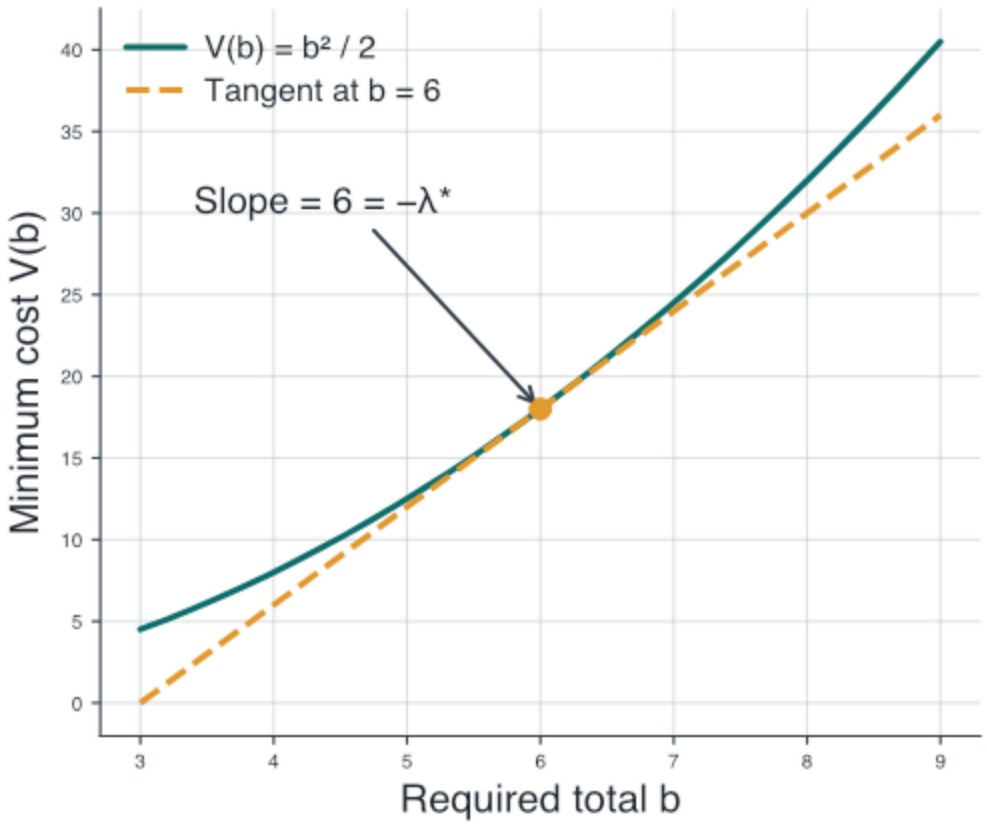
Let $x+y = b$ and $V(b) = \min(x^2+y^2)$.

$L = x^2+y^2 + \lambda(x+y-b)$.

$x^* = y^* = b/2$; $V(b) = b^2/2$; $\lambda^* = -b$.

$V'(b) = -\lambda^*$. At $b = 6$: $V' = 6$.

For $\Delta b = 0.1$: $\Delta V \approx 6(0.1) = 0.6$.
Exact change: $V(6.1) - V(6) = 0.605$.



The multiplier gives a local rate of change, not an exact finite change.

Worked example: minimum energy

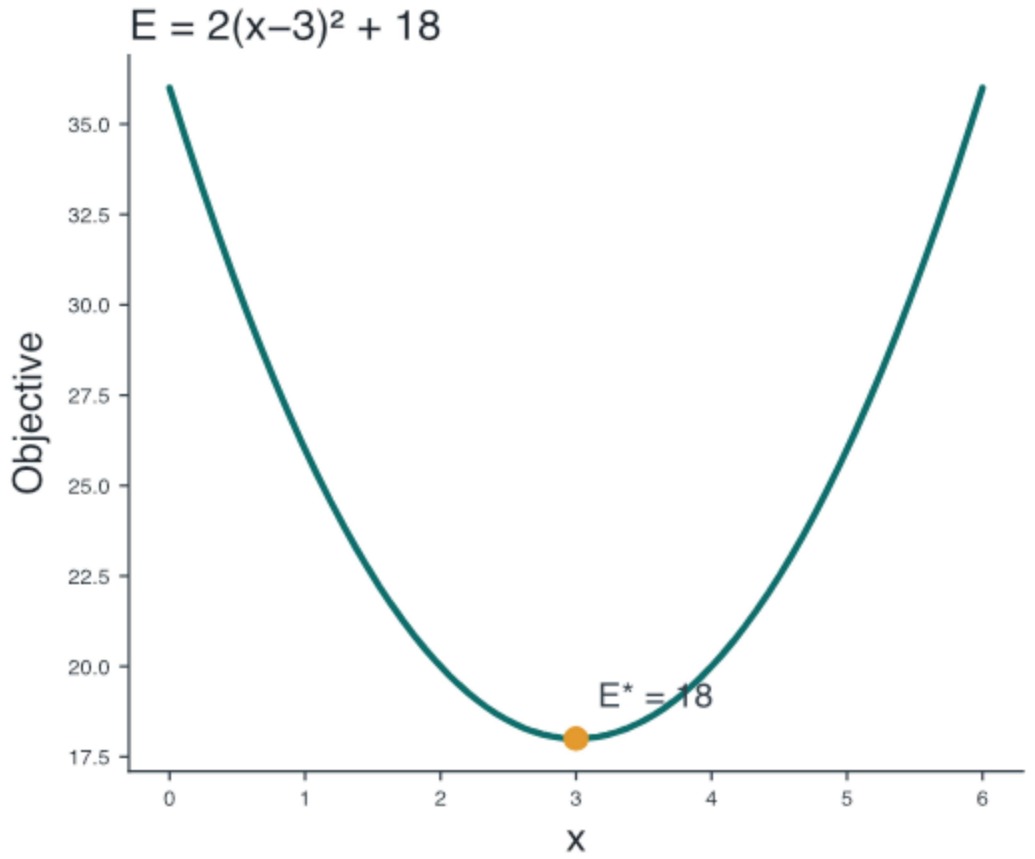
Minimize $E(x,y) = x^2 + y^2$ subject to $x + y = 6$.

$L = x^2 + y^2 + \lambda(x + y - 6)$

$2x + \lambda = 0,\ \ 2y + \lambda = 0,\ \ x + y = 6$

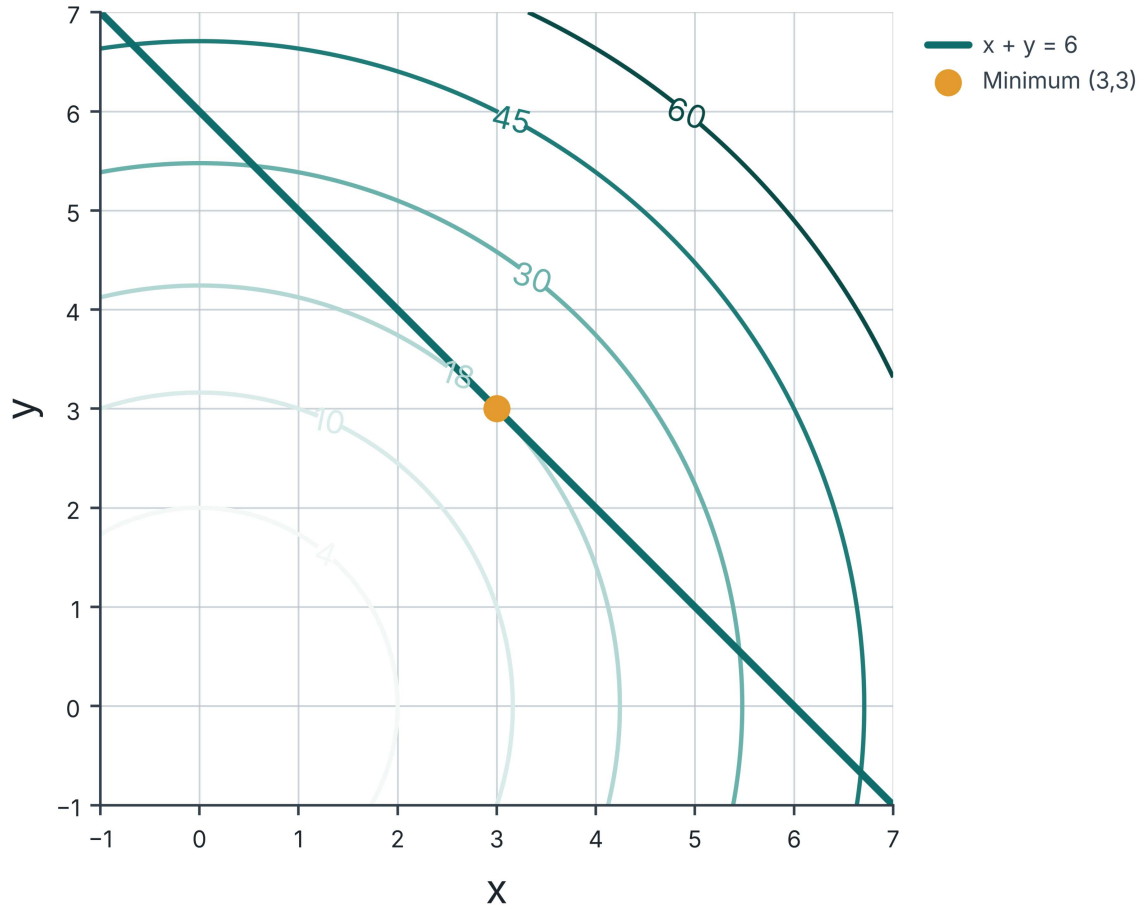
$x^* = y^* = 3,\ \ \lambda^* = -6,\ \ E^* = 18$

$E(x,6-x) = 2(x-3)^2 + 18$ proves a global minimum.

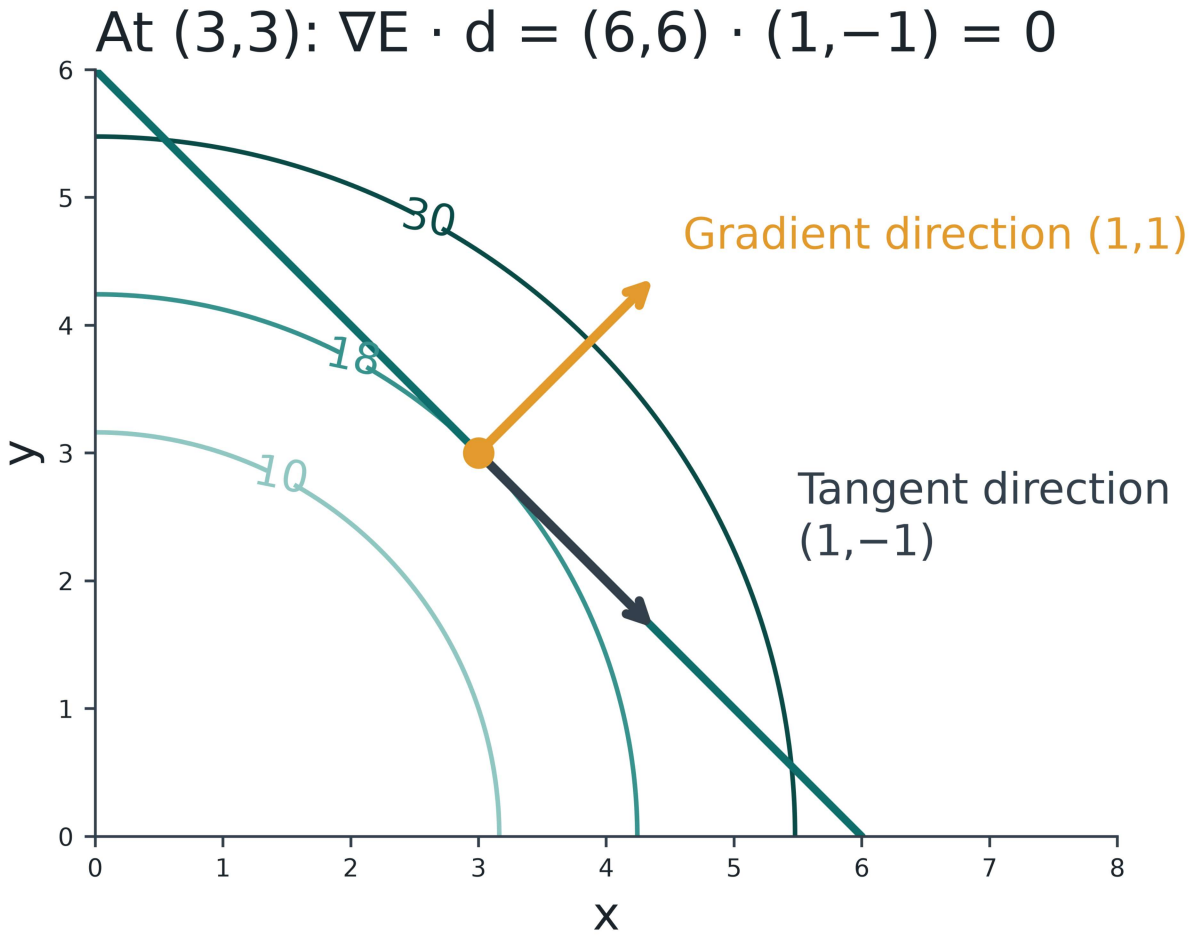


The reduced objective proves global optimality.

Equality geometry



The gradient is perpendicular to feasible motion



Finding both a maximum and a minimum

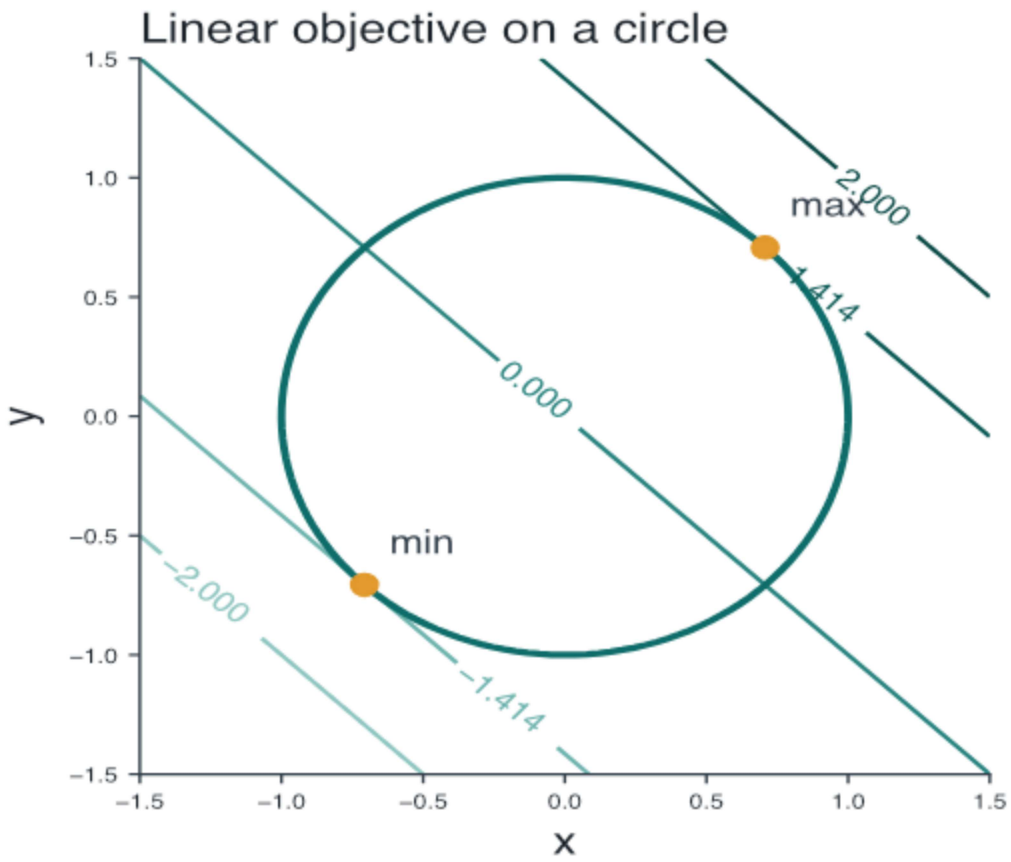
Optimize $f(x,y) = x + y$ subject to $x^2 + y^2 = 1$.

$L = x + y + \lambda(x^2 + y^2 - 1)$

$1 + 2\lambda x = 0,\ \ 1 + 2\lambda y = 0 \rightarrow \lambda \neq 0$ and $x = y$

Candidates: $(1/\sqrt{2}, 1/\sqrt{2})$ and $(-1/\sqrt{2}, -1/\sqrt{2})$

Maximum = $\sqrt{2}$. Minimum = $-\sqrt{2}$.



The two tangent level lines give the maximum and minimum.

Practice: fixed total production

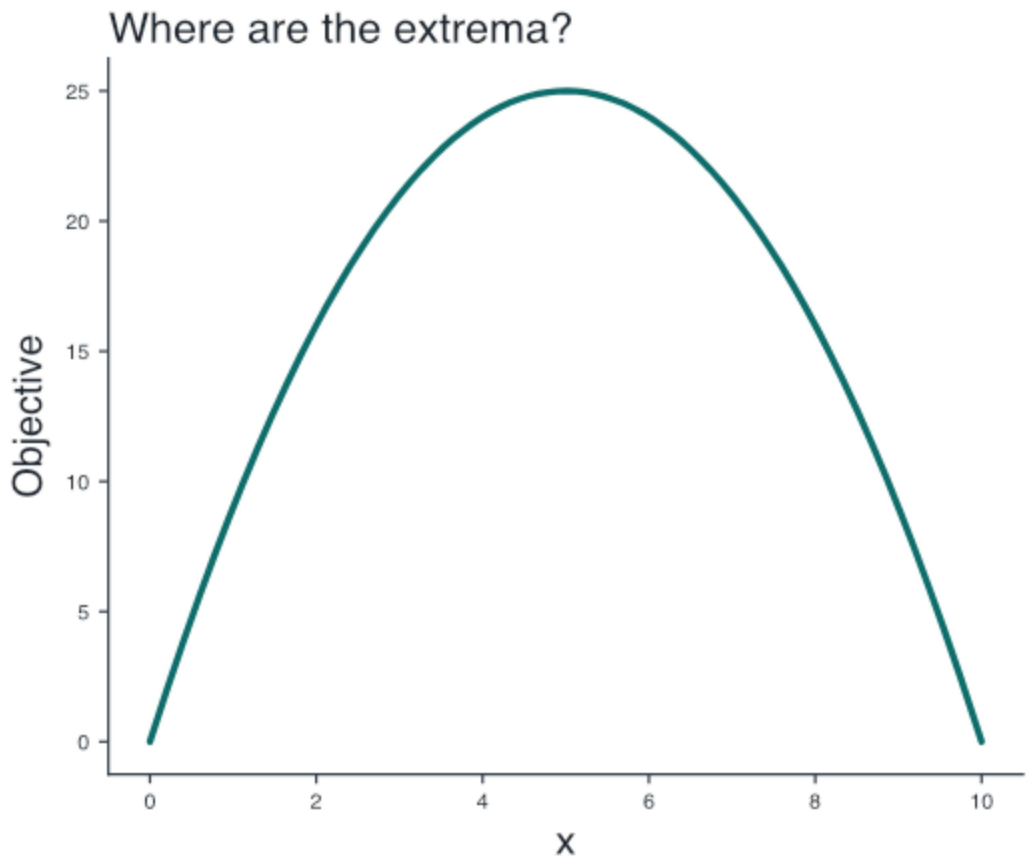
Two nonnegative rates satisfy x + y = 10.

Find both the maximum and minimum of P(x,y) = xy.

First examine the interior using Lagrange multipliers.

Then inspect the endpoints of the feasible segment.

Explain why an interior calculation alone is incomplete.



Compare the interior with both endpoints of the feasible segment.

Fixed total production: solution

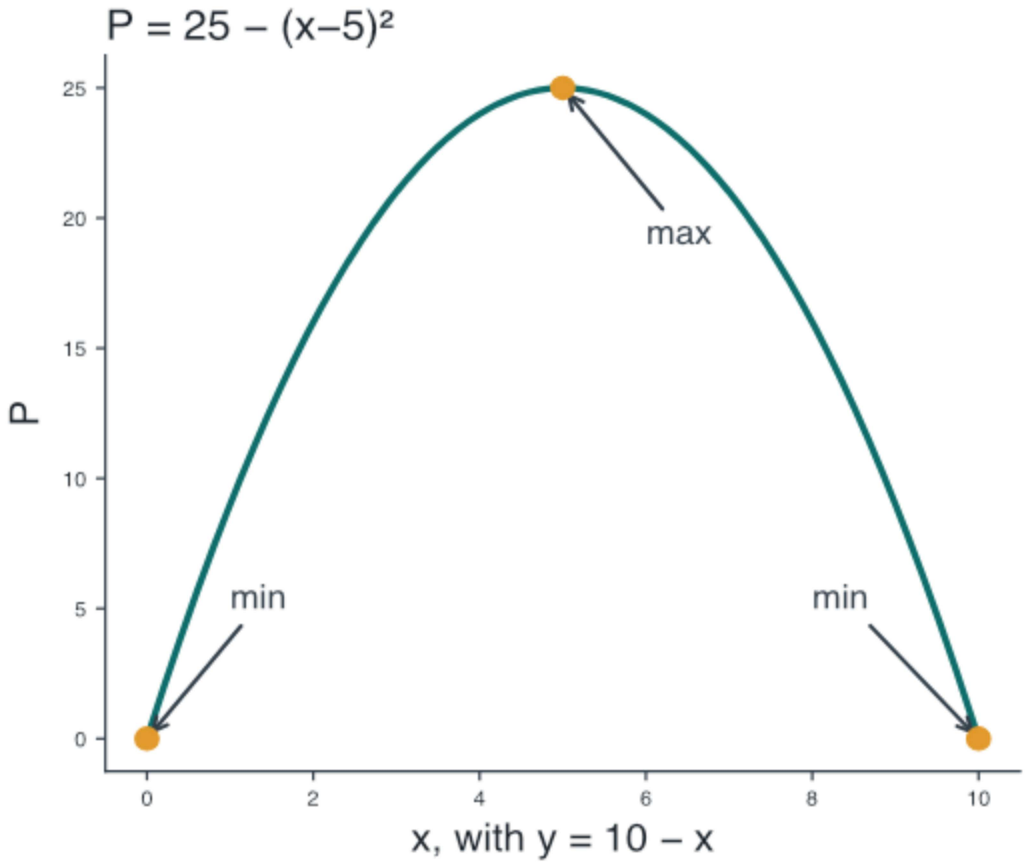
Interior: $y + \lambda = 0$, $x + \lambda = 0$, $x + y = 10$.

$x = y = 5 \rightarrow P = 25$.

$P(x,10-x) = 25 - (x-5)^2$ for $0 \le x \le 10$.

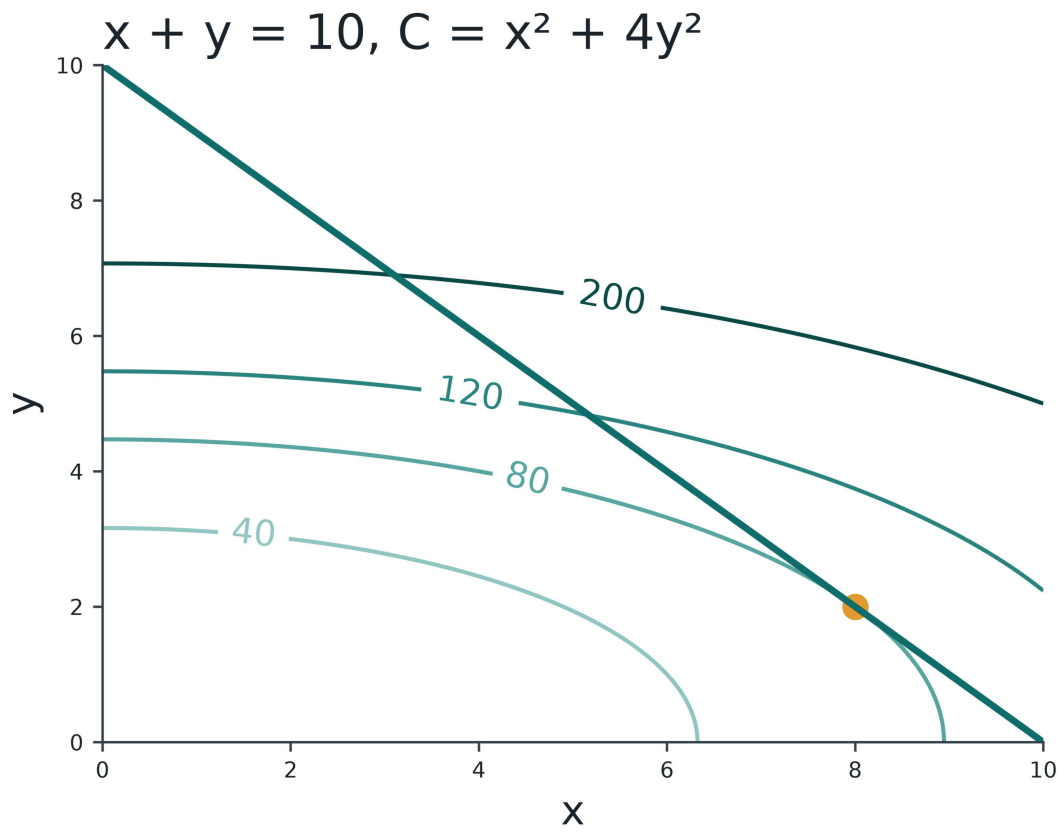
Global maximum: 25 at (5,5).

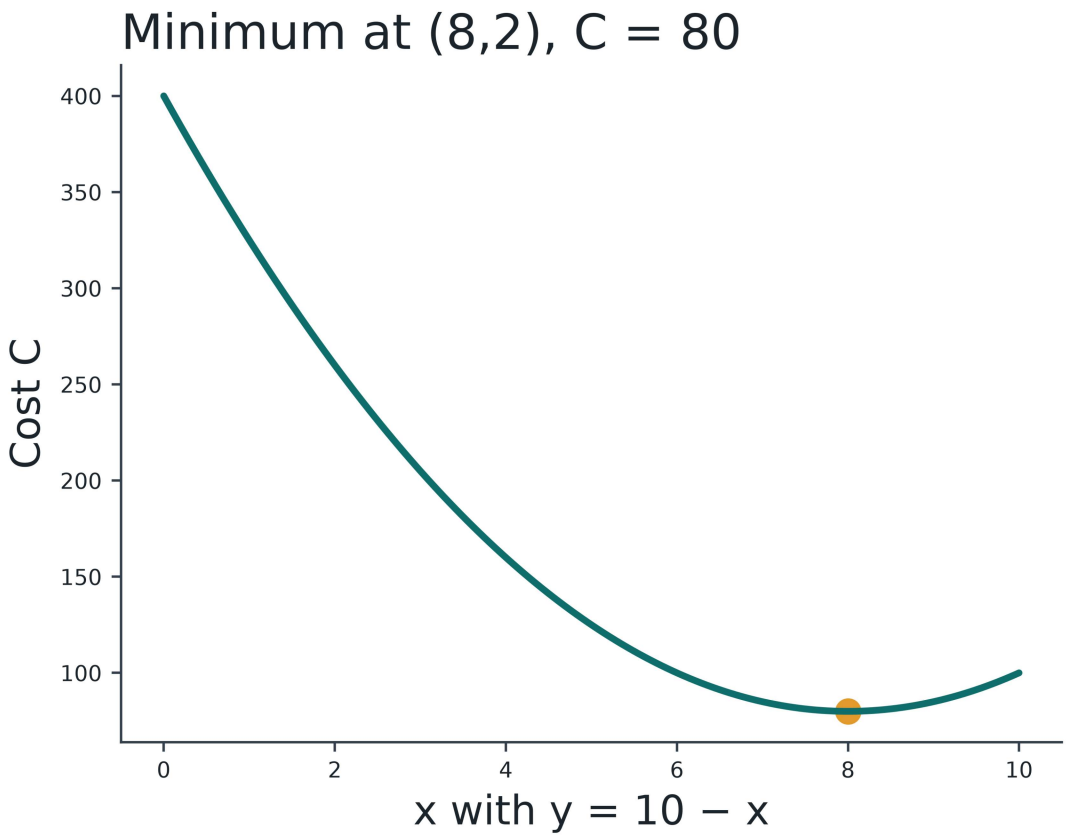
Global minimum: 0 at (0,10) and (10,0).



Interior maximum and boundary minima coexist.

Lagrange example: sharing a production load





Minimize $C = x^2 + 4y^2$ subject to $x+y = 10$, $x,y \geq 0$.

Sharing a production load: solution

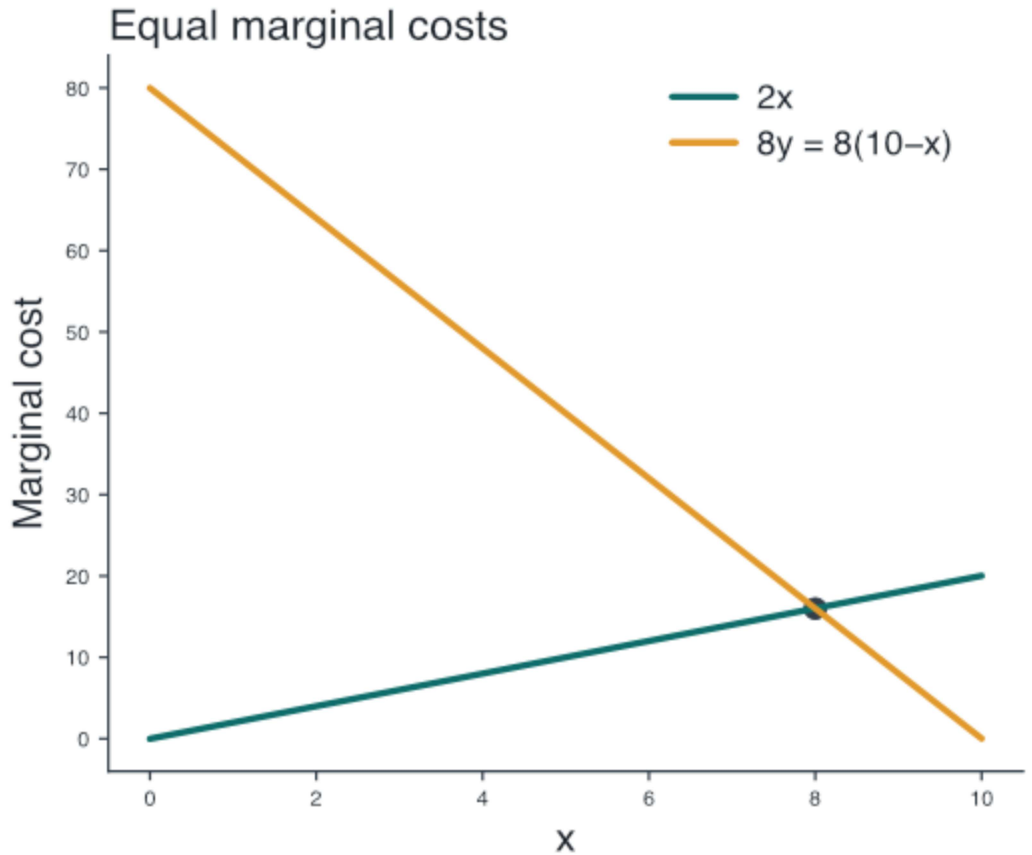
Minimize $C = x^2 + 4y^2$, with $x+y = 10$ and $x,y \geq 0$.

$L = x^2 + 4y^2 + \lambda(x+y-10)$

$2x+\lambda = 0,\ \ 8y+\lambda = 0 \rightarrow x = 4y$

$x^* = 8,\ y^* = 2,\ \lambda^* = -16,\ C^* = 80$

$C(x,10-x) = 5(x-8)^2 + 80 \rightarrow$ unique global minimum.



At x = 8 and y = 2, both marginal costs equal 16.

Break

10 minutes

After the break: active constraints and KKT

A consistent inequality convention

Minimize $f(x)$

subject to $g_i(x) \le 0$ and $h_j(x) = 0$

$L = f + \Sigma \mu_i g_i + \Sigma \lambda_j h_j$

Use $\mu_i \ge 0$. Equality multipliers remain unrestricted.

To maximize P, minimize $f = -P$.

Maximum profit

maximize P

↓

Change objective

minimize −P

Minimum delivery

$x + y \ge 4$

↓

Standard constraint

$g = 4 - x - y \le 0$

The four KKT conditions

Stationarity: $\nabla f + \Sigma\, \mu_i \nabla g_i + \Sigma\, \lambda_j \nabla h_j = 0$

Primal feasibility: $g_i \le 0$ and $h_j = 0$

Dual feasibility: $\mu_i \ge 0$

Complementary slackness: $\mu_i g_i = 0$ for each i

At x = 2, μ = 2

$f = (x-3)^2$, $g = x-2$

Stationarity

$-2 + 2 = 0$

Feasibility

$g = 0 \le 0$, $\mu = 2 \ge 0$

Complementarity

$\mu g = 2 \times 0 = 0$

One example for all four KKT conditions

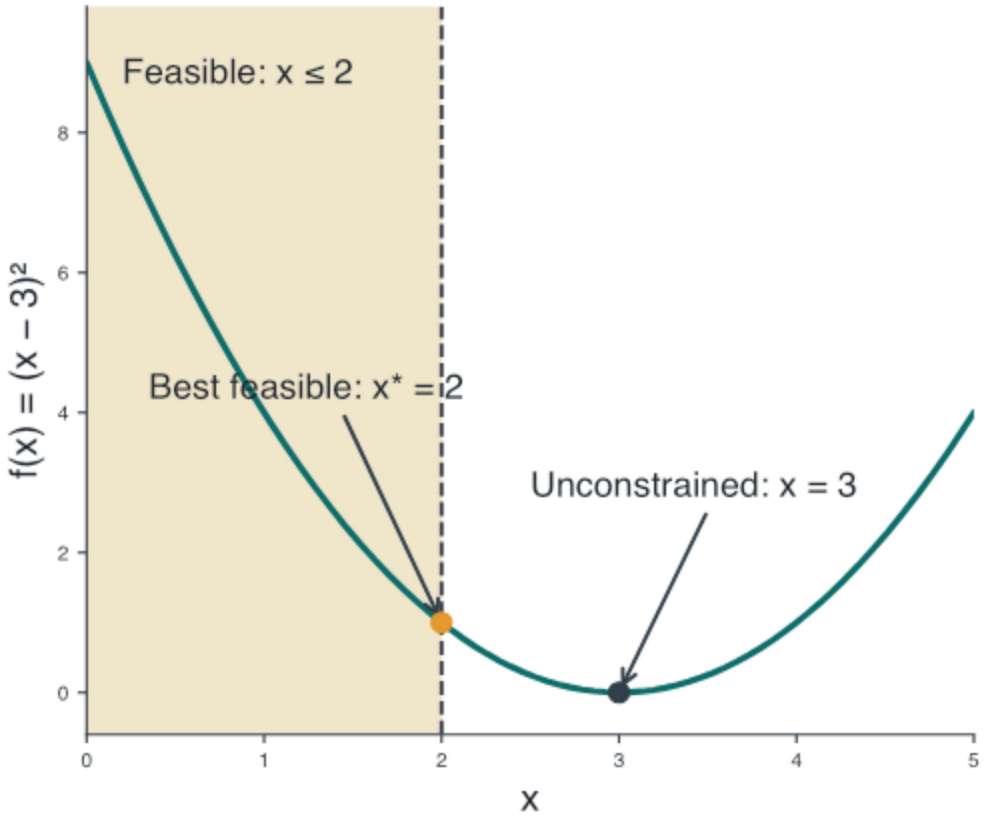
Minimize $f(x) = (x-3)^2$ subject to $x \le 2$.

Write $g(x) = x-2 \le 0$.

$L(x, \mu) = (x-3)^2 + \mu(x-2)$.

Without the bound: $x = 3$.
With the bound: the best feasible point is $x^* = 2$.

At $x^* = 2$, all four conditions hold with $\mu^* = 2$.



The unconstrained minimum is excluded by the upper bound.

Primal feasibility: is the point allowed?

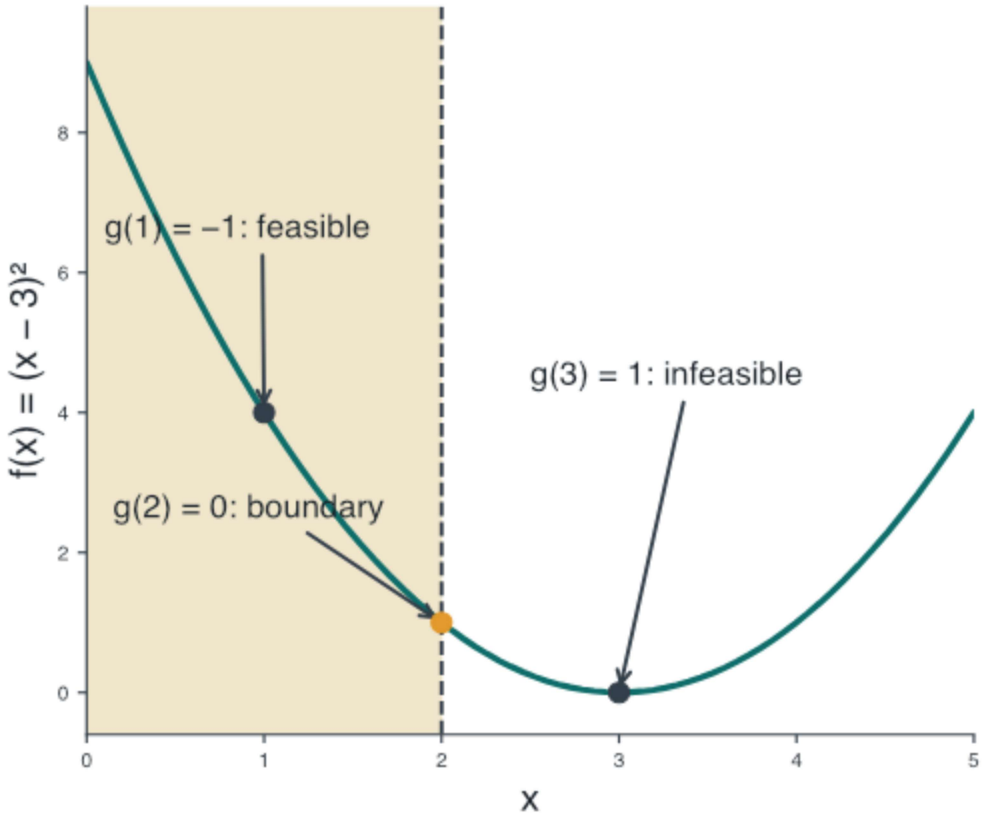
All $g_i(x^*) \le 0$ and all $h_j(x^*) = 0$.

The candidate must satisfy every original constraint.

Here: g(2) = 2−2 = 0, so x = 2 is allowed.

x = 3 gives a lower cost, but g(3) = 1 > 0.
It is infeasible and must be rejected.

Feasibility alone does not prove optimality:
x = 1 is feasible, but its cost is higher.



Check the original constraints before comparing objective values.

Stationarity: which gradients balance?

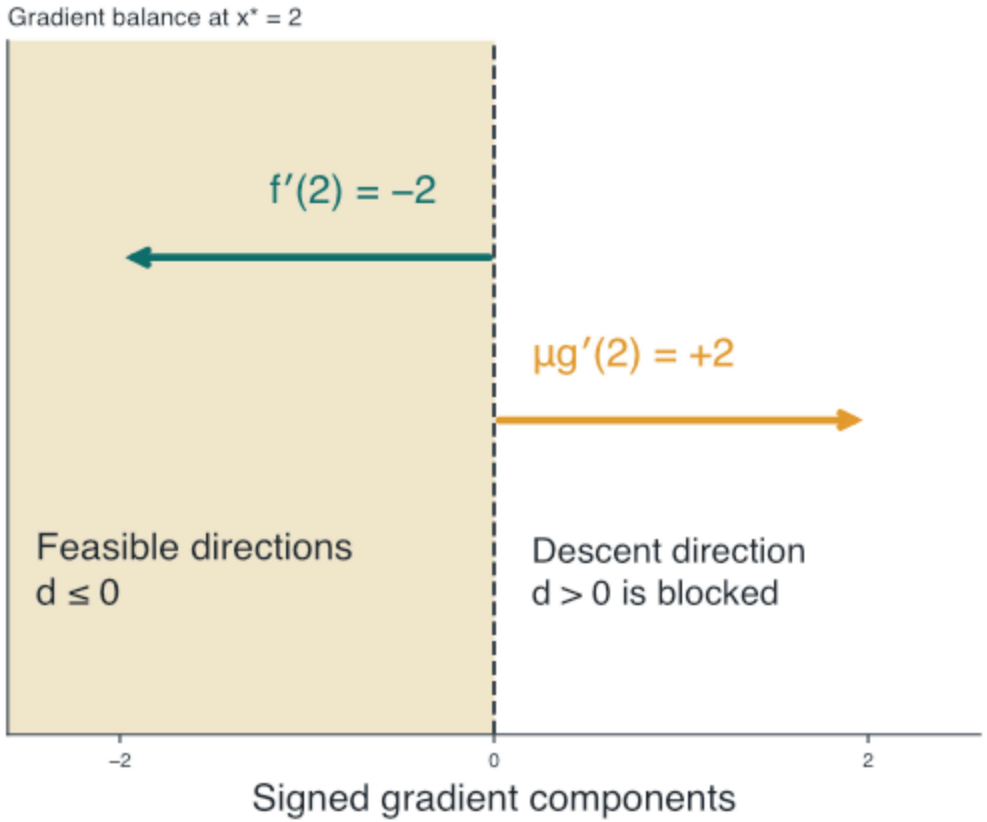
$\nabla f + \Sigma \mu_i \nabla g_i + \Sigma \lambda_j \nabla h_j = 0.$

The objective gradient is balanced by weighted constraint gradients.

Here: $2(x-3) + \mu = 0$.

At $x = 2$: $-2 + \mu = 0$, hence $\mu = 2$.

$f'(2)$ is not zero. Moving right lowers f, but violates $x \le 2$.



For a feasible direction $d \le 0$, $f'(2)d = -2d \ge 0$.

Dual feasibility: why must μ be nonnegative?

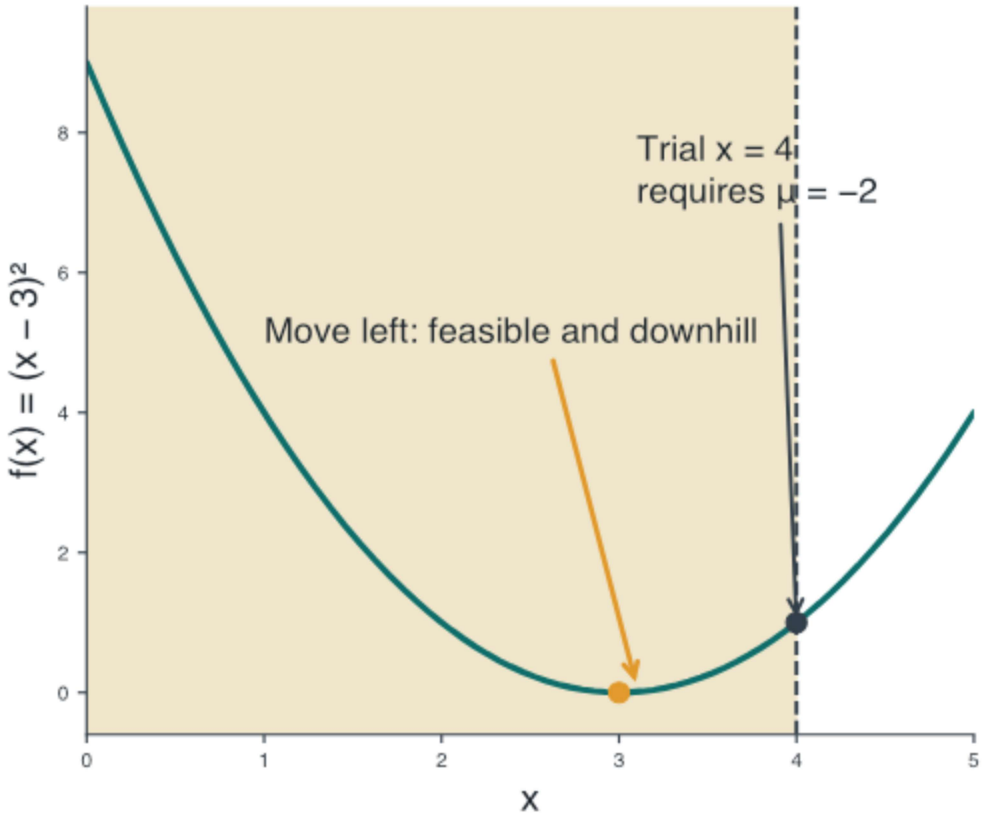
For min f with $g_i \le 0$: $\mu_i \ge 0$.
Equality multipliers λ_j remain unrestricted.

The active constraint gradient must balance ∇f with an allowed sign.

Change the bound to $x \le 4$; try $x = 4$.

Stationarity gives $2(4-3) + \mu = 0$,
so $\mu = -2$. Reject this candidate.

Moving left stays feasible and lowers f.
The true solution is $x^* = 3$, $\mu^* = 0$.

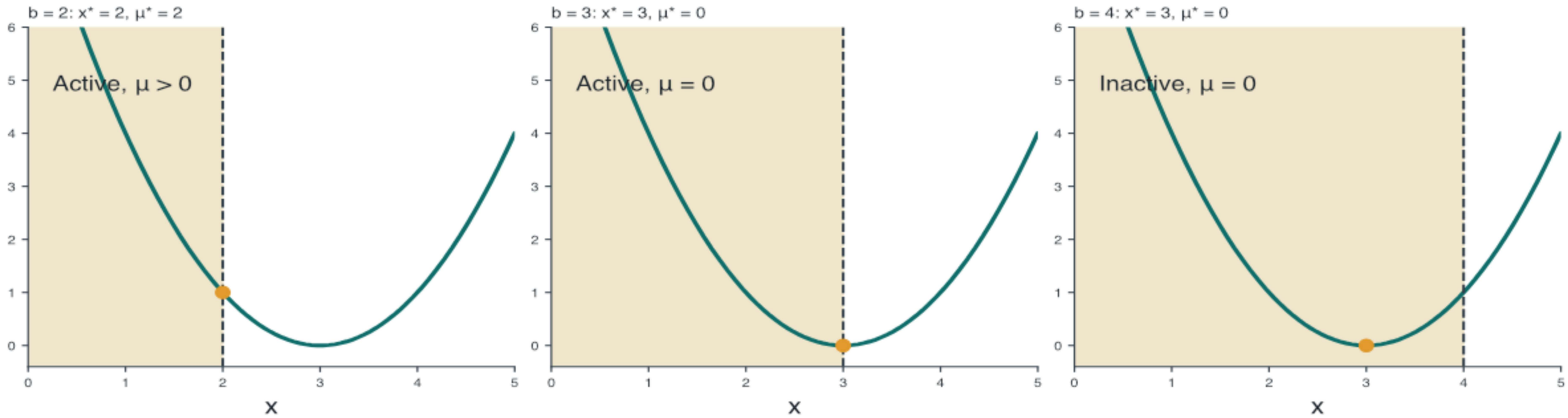


Here, μ = −2 rejects a point with a feasible downhill direction.

Complementary slackness: when does a bound matter?

$\mu_i g_i(x^*) = 0$: either the constraint is active, or its multiplier is zero.

If $g_i < 0$, then $\mu_i = 0$. If $\mu_i > 0$, then $g_i = 0$. An active constraint can have $\mu_i = 0$.



For $f = (x-3)^2$ and $x \le b$: $\mu^* g(x^*)$ equals 2×0, 0×0, and $0\times(-1)$, respectively.

All four conditions: one complete check

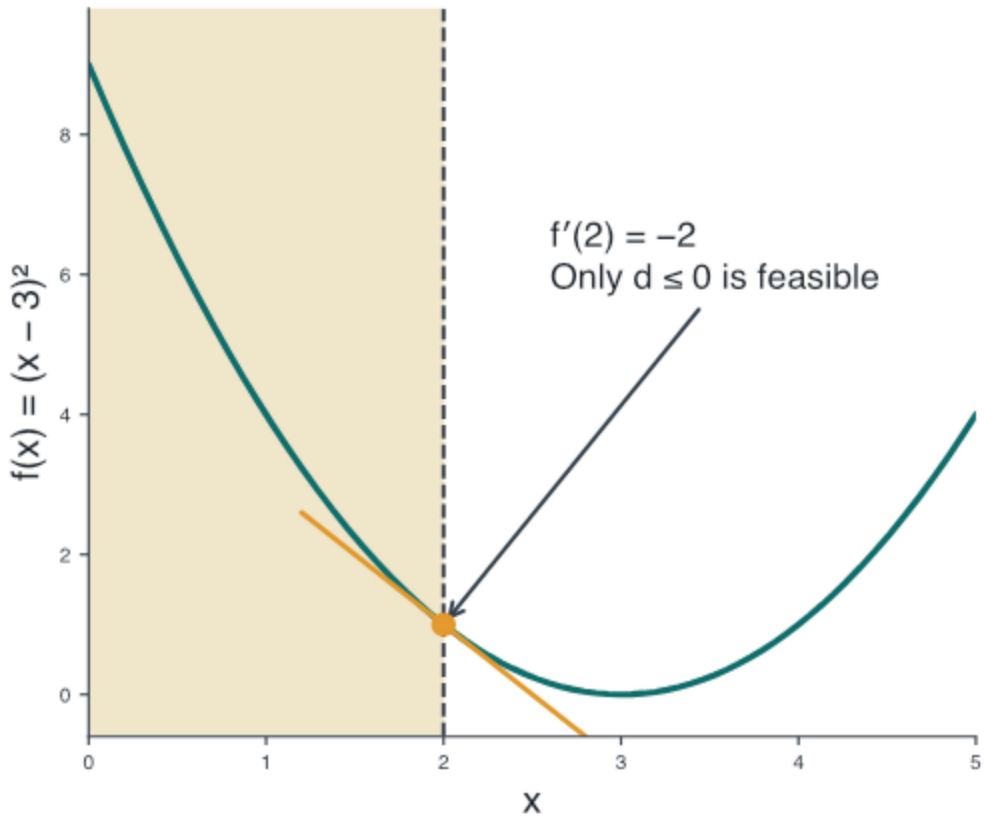
Candidate: $x^* = 2$ and $\mu^* = 2$.

Primal feasibility: $g = 2-2 = 0 \le 0$.
Dual feasibility: $\mu = 2 \ge 0$.

Stationarity: $2(2-3) + 2 = 0$.

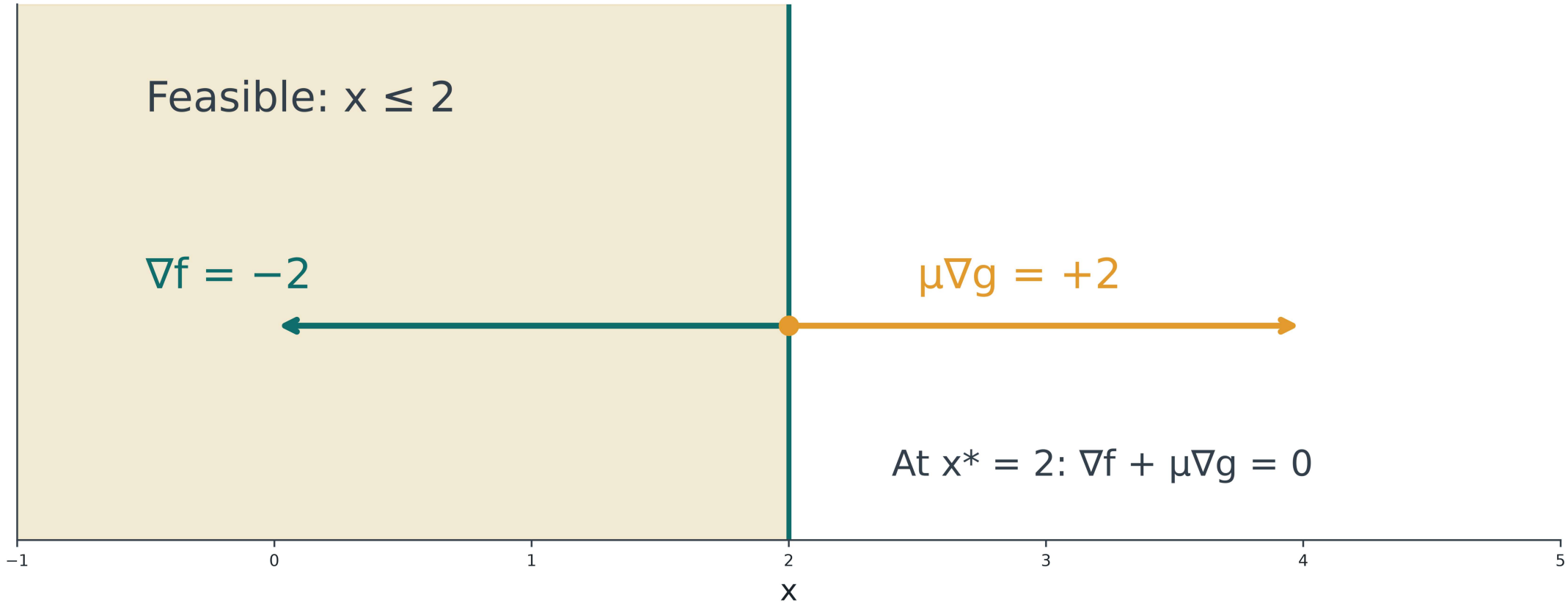
Complementary slackness: $\mu g = 2 \times 0 = 0$.

f is strictly convex and g is affine.
This KKT point is the unique global minimum.



Minimum value f(2) = 1. For nonconvex problems, KKT alone is not a global certificate.

KKT balances the objective and constraint gradients



Minimize f = (x−3)² subject to g = x−2 ≤ 0. At x* = 2, μ = 2.

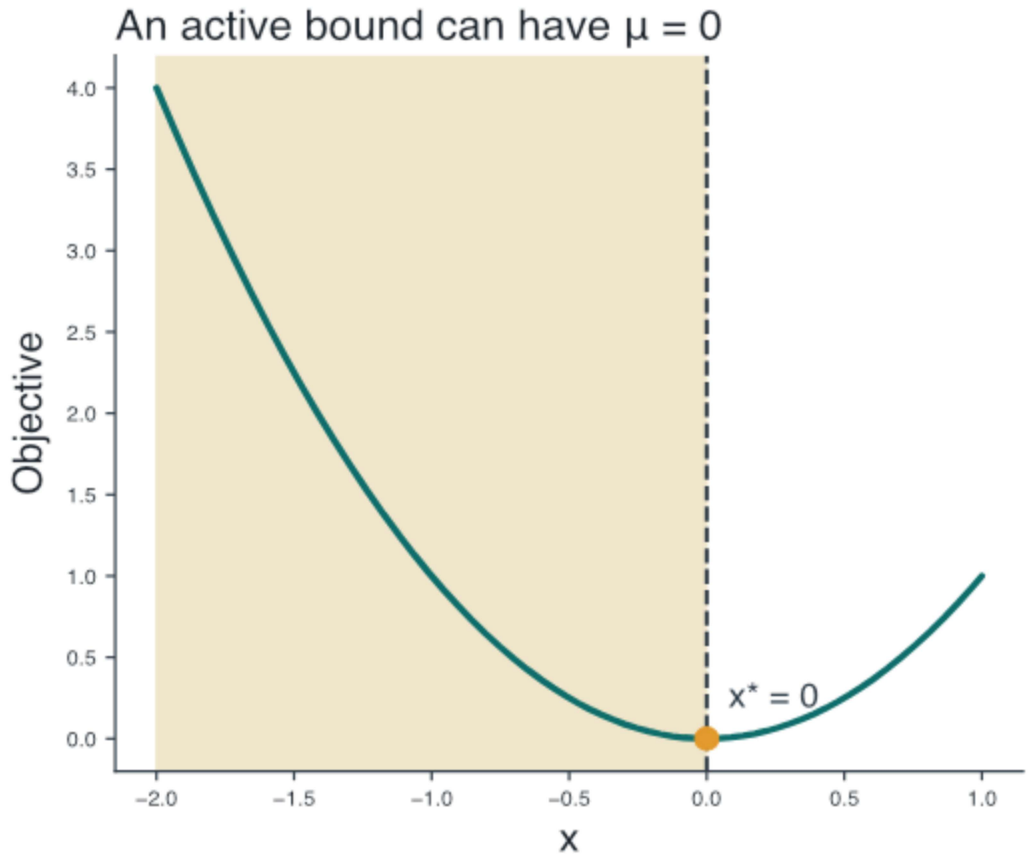
Active and inactive constraints

Inactive: $g_i(x^*) < 0 \rightarrow \mu_i = 0$.

Active: $g_i(x^*) = 0 \rightarrow \mu_i$ may be zero or positive.

$\mu_i > 0 \rightarrow$ the constraint is active.

Example: min x^2 subject to $x \leq 0$ has $x^* = 0$, $\mu^* = 0$.



At $x^* = 0$, the constraint is active but the objective gradient is already zero.

Worked example: an active upper bound

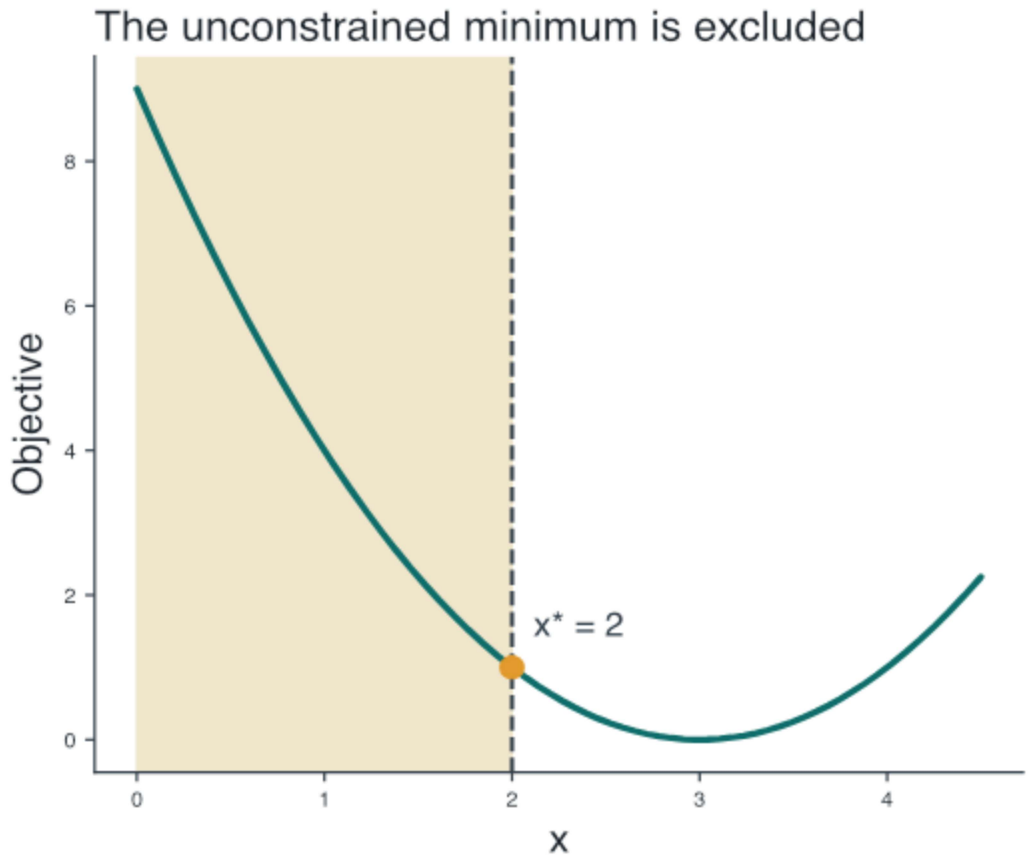
Minimize $f(x) = (x-3)^2$ subject to $x \le 2$.

$L = (x-3)^2 + \mu(x-2)$

Stationarity: $2(x-3) + \mu = 0$

Inactive case: $\mu = 0 \rightarrow x = 3$, which is infeasible.

Active case: $x = 2 \rightarrow \mu = 2 \ge 0$. Minimum $f^* = 1$.



The feasible minimum lies on the bound $x = 2$.

Worked example: an inactive upper bound

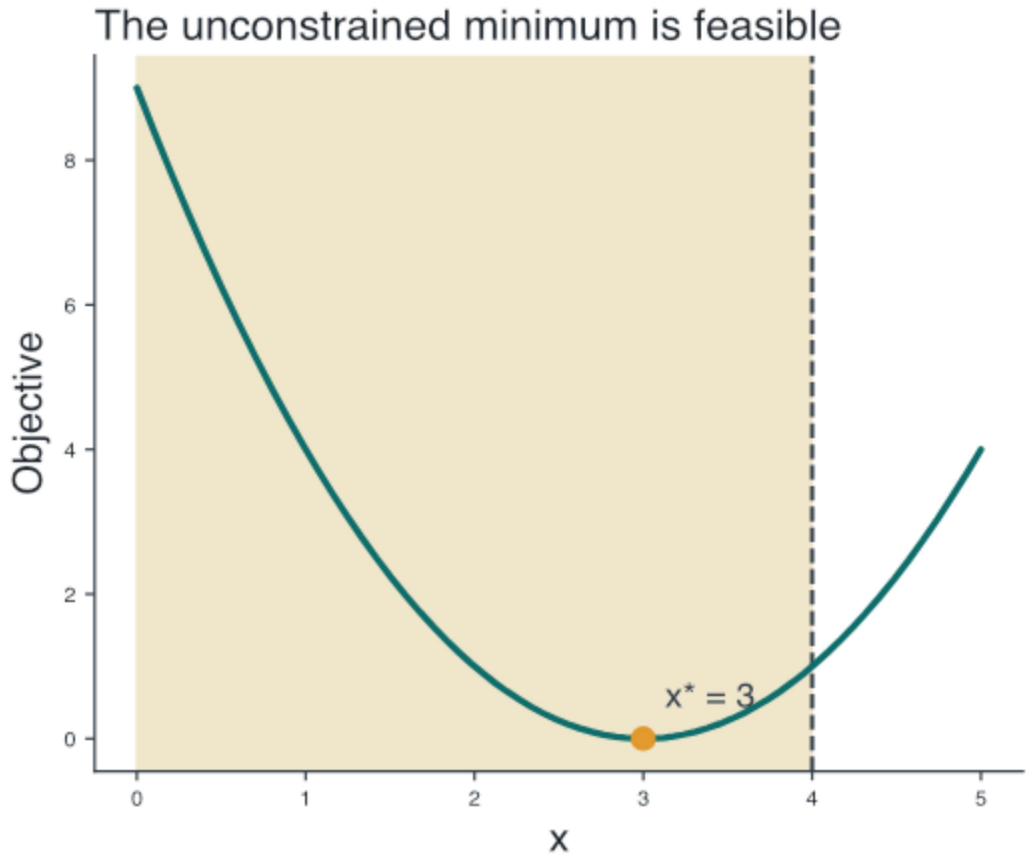
Change the bound to $x \le 4$.

Inactive case: $\mu = 0 \rightarrow x = 3 < 4$.

Active case: $x = 4 \rightarrow \mu = -2$, which is inadmissible.

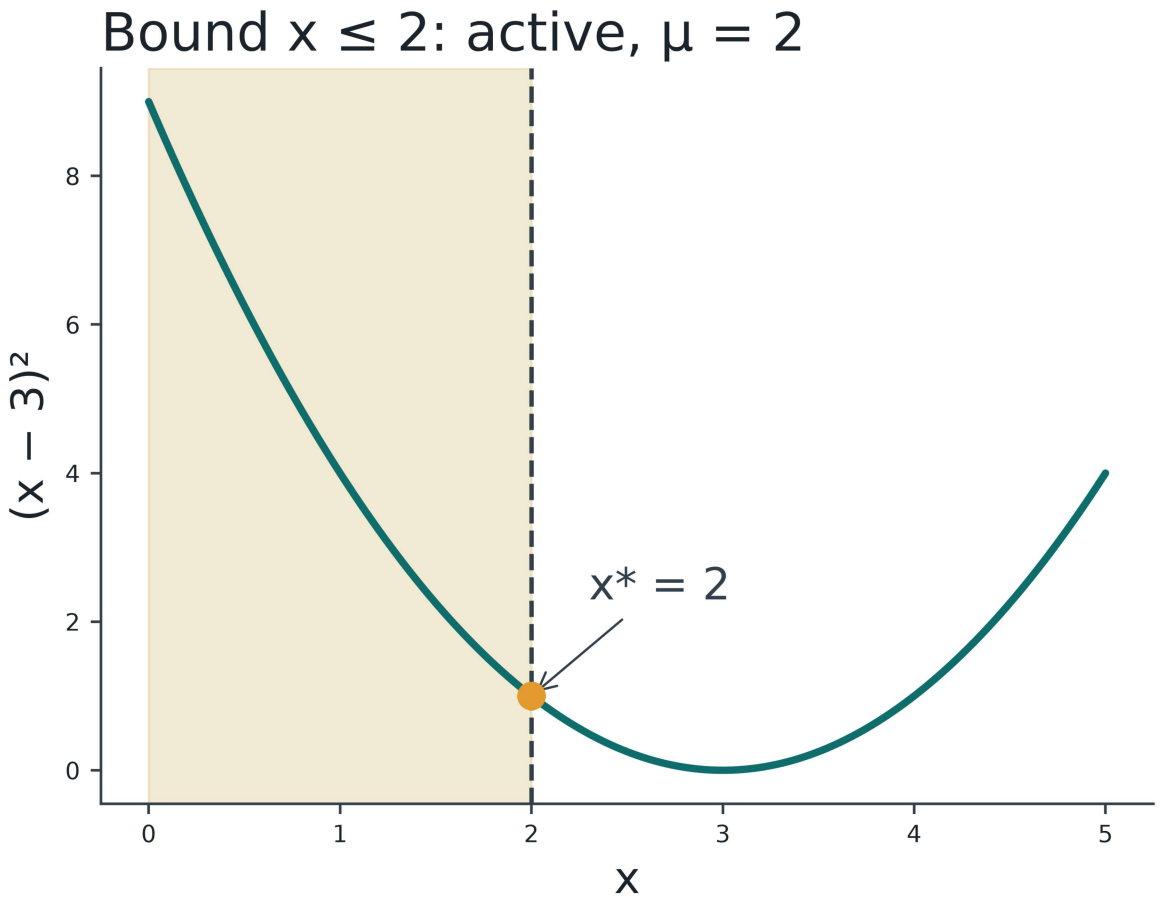
Solution: $x^* = 3$, $\mu^* = 0$, $f^* = 0$.

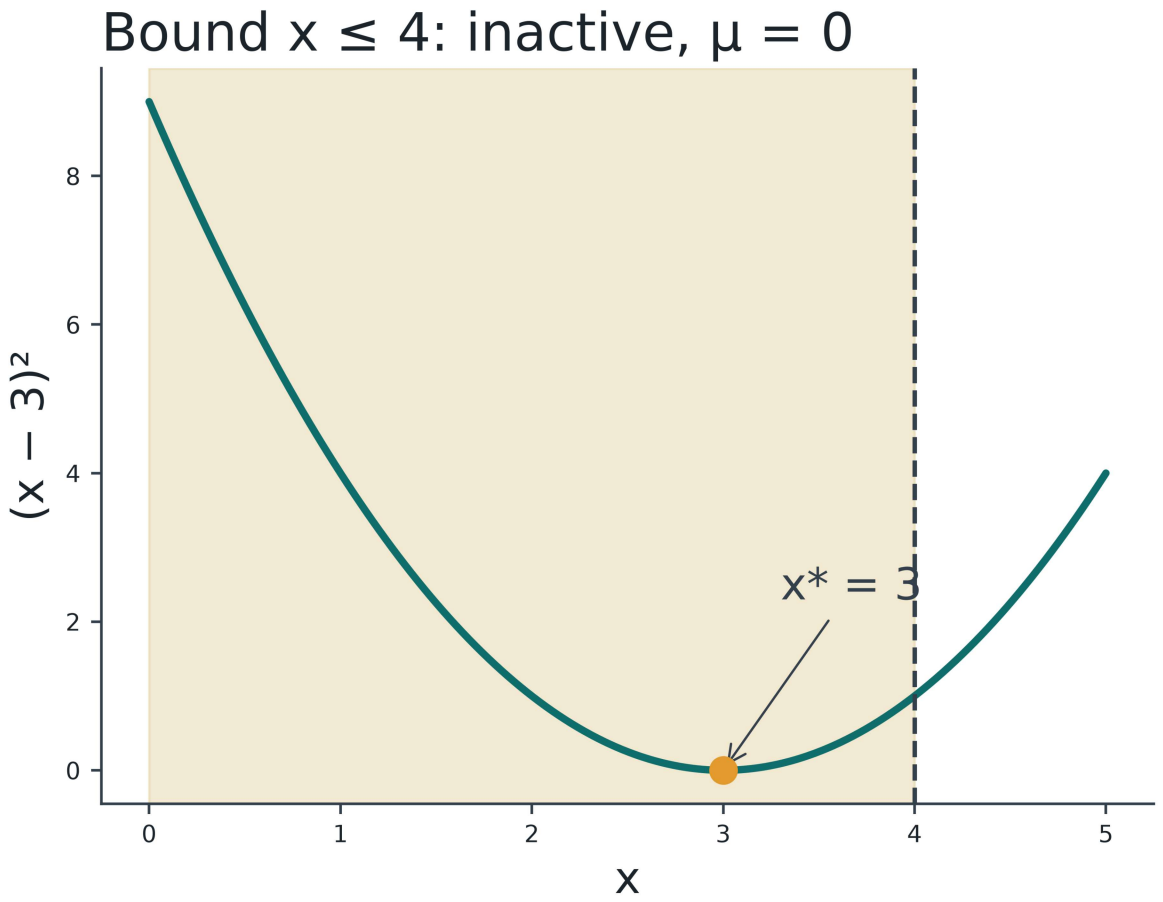
Always reject negative inequality multipliers.



At $x^* = 3 < 4$, the bound is inactive and $\mu = 0$.

An upper bound can be active or inactive





A practical active-set procedure

1. Rewrite the problem as min f with $g_i \le 0$.
2. Choose a candidate set of active inequalities.
3. Set active $g_i = 0$ and inactive $\mu_i = 0$.
4. Solve stationarity and equality constraints.
5. Reject infeasible points or $\mu_i < 0$, then certify.

Choose a branch

$\min (x-3)^2,\ x \le 2$

↓

Inactive trial: $\mu = 0$

$x = 3 \rightarrow$ infeasible

↓

Active trial: $x = 2$

$\mu = 2 \rightarrow$ admissible

↓

Verify and certify

All KKT + convexity

When KKT is necessary or sufficient

Necessity: a local minimum + a constraint qualification.

LICQ: equality and active inequality gradients are independent.

Sufficiency: convex f and g_i, affine h_j, and a KKT point.

Then the point is a global minimum.

Nonconvex problems need additional classification.

Local minimum + CQ

KKT is necessary

KKT + convex problem

Global minimum

Strict convexity

At most one minimizer

Nonconvex problem

KKT alone is insufficient

A constraint qualification can fail

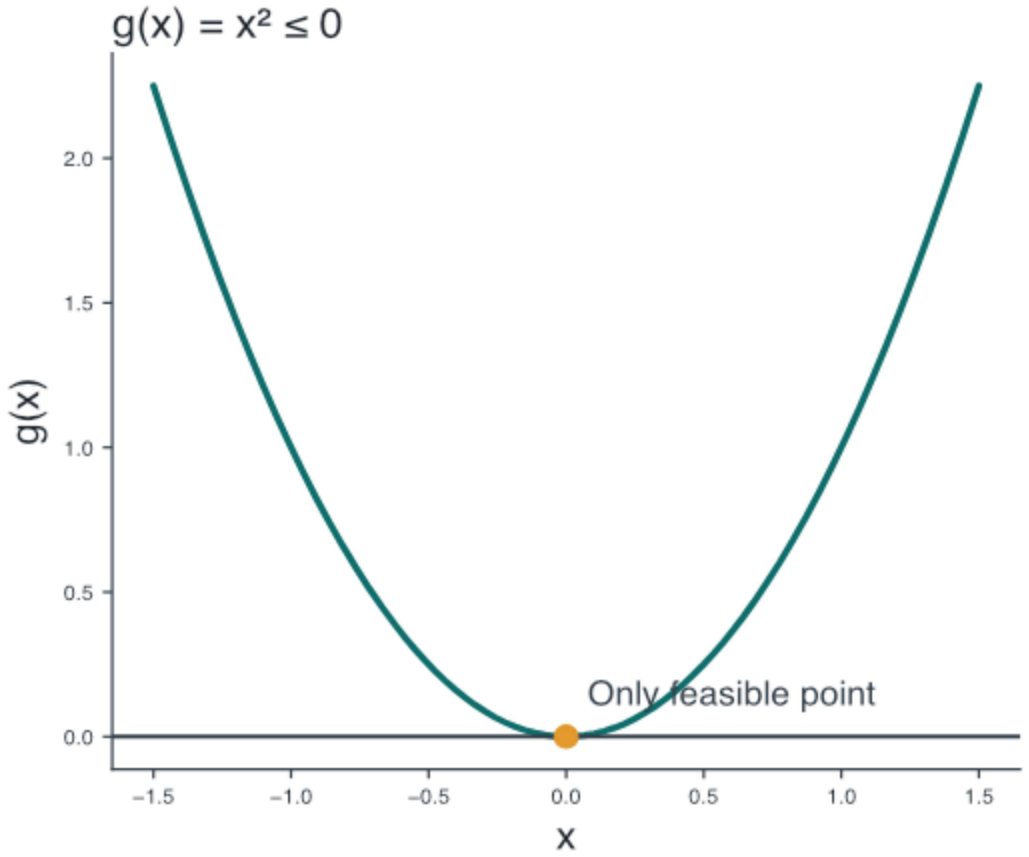
Minimize $f(x) = x$ subject to $g(x) = x^2 \leq 0$.

The only feasible point is $x^* = 0$, so it is optimal.

But $\nabla g(0) = 0$.

KKT stationarity would require $1 + \mu \cdot 0 = 0$.

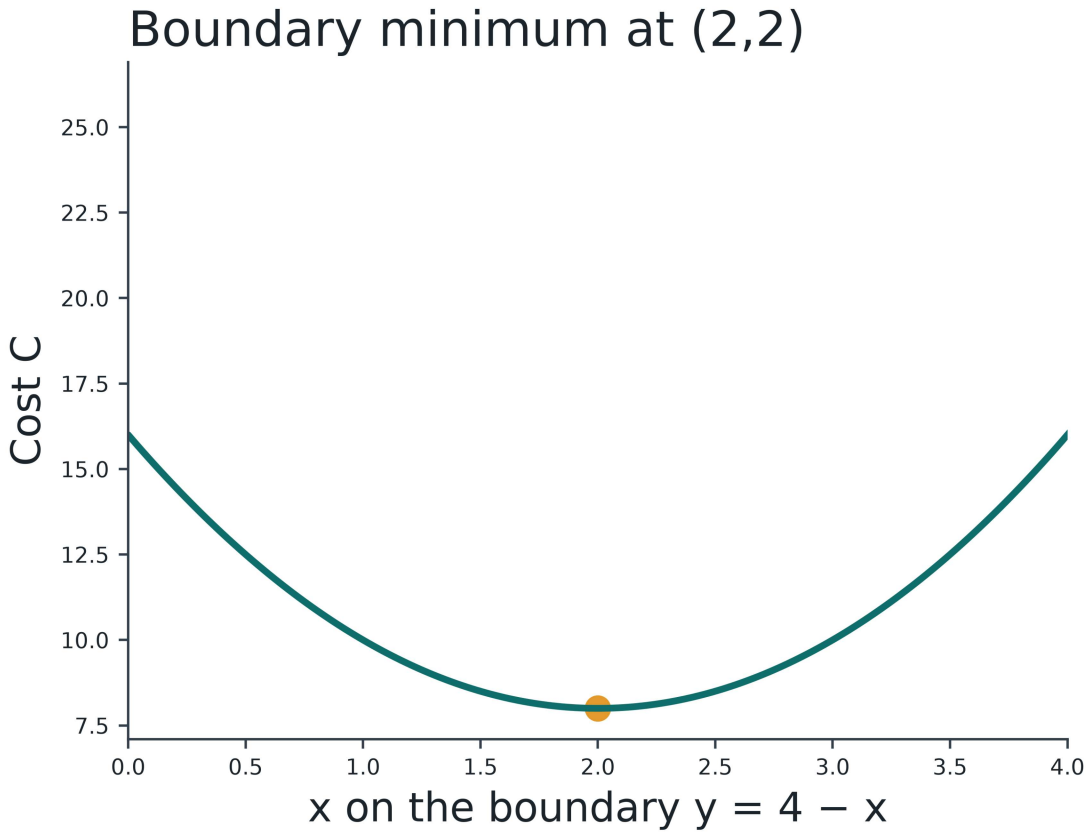
No multiplier works. Optimality does not always imply KKT.



The constraint gradient vanishes at the only feasible point.

KKT example: meeting a minimum delivery target





Minimize $C = x^2 + y^2$ subject to $x+y \geq 4$, $x,y \geq 0$.

Minimum delivery: KKT solution

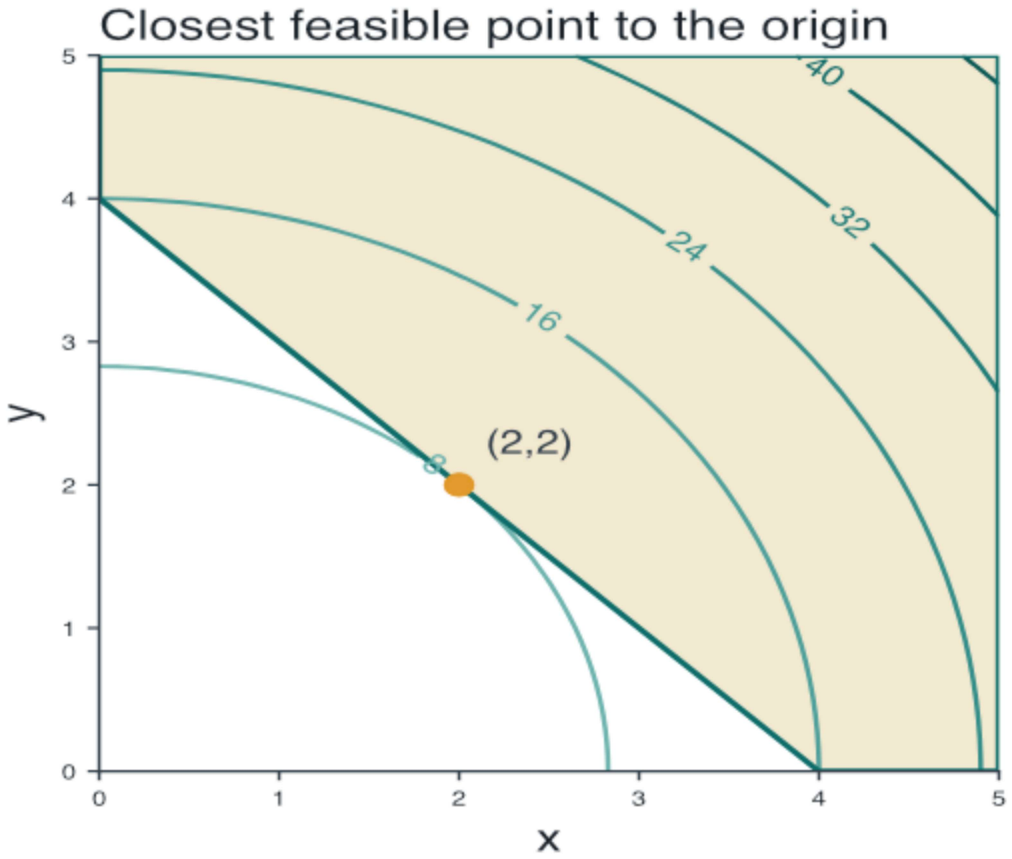
$L = x^2+y^2 + \mu(4-x-y) - \alpha x - \beta y$

Stationarity: $2x-\mu-\alpha = 0,\ \ 2y-\mu-\beta = 0$

Trial: $x+y = 4,\ x,y > 0 \rightarrow \alpha = \beta = 0$

$x^* = y^* = 2,\ \mu^* = 4,\ C^* = 8$

Feasible KKT point + strict convexity → unique global minimum.



Rewrite $x+y \geq 4$ as $4-x-y \leq 0$ before forming L.

Process application: shared capacity

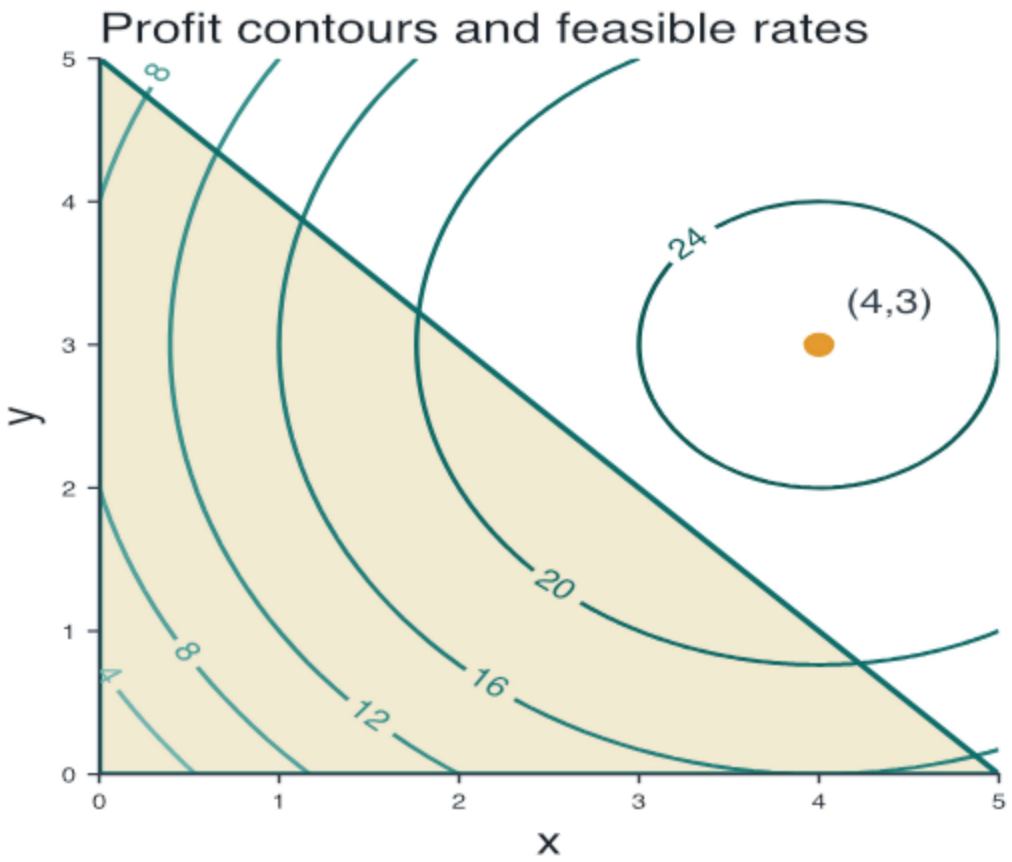
Choose nonnegative normalized production rates x and y.

Profit index: $P = 8x + 6y - x^2 - y^2$

Shared capacity: $x + y \le 5$

Find the profit-maximizing operating point.

Illustrative model with diminishing marginal returns.



The unconstrained point (4,3) lies outside the capacity triangle.

Resource allocation: full KKT system

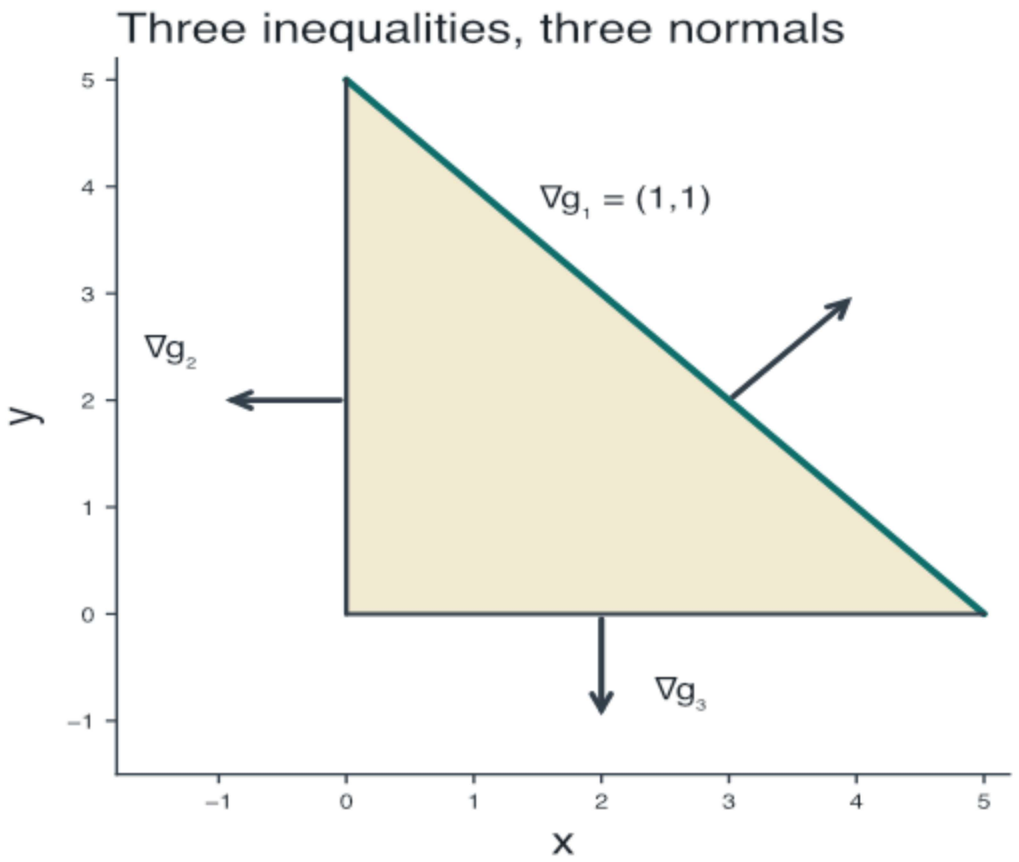
$f = -P = x^2 + y^2 - 8x - 6y$

$L = f + \mu(x+y-5) - \alpha x - \beta y$

$2x-8+\mu-\alpha = 0, \quad 2y-6+\mu-\beta = 0$

$\mu(x+y-5) = 0, \quad \alpha x = 0, \quad \beta y = 0$

$x,y \geq 0, \quad x+y \leq 5, \quad \mu,\alpha,\beta \geq 0$



The lower bounds also belong in the full KKT system.

Resource allocation: solve and verify

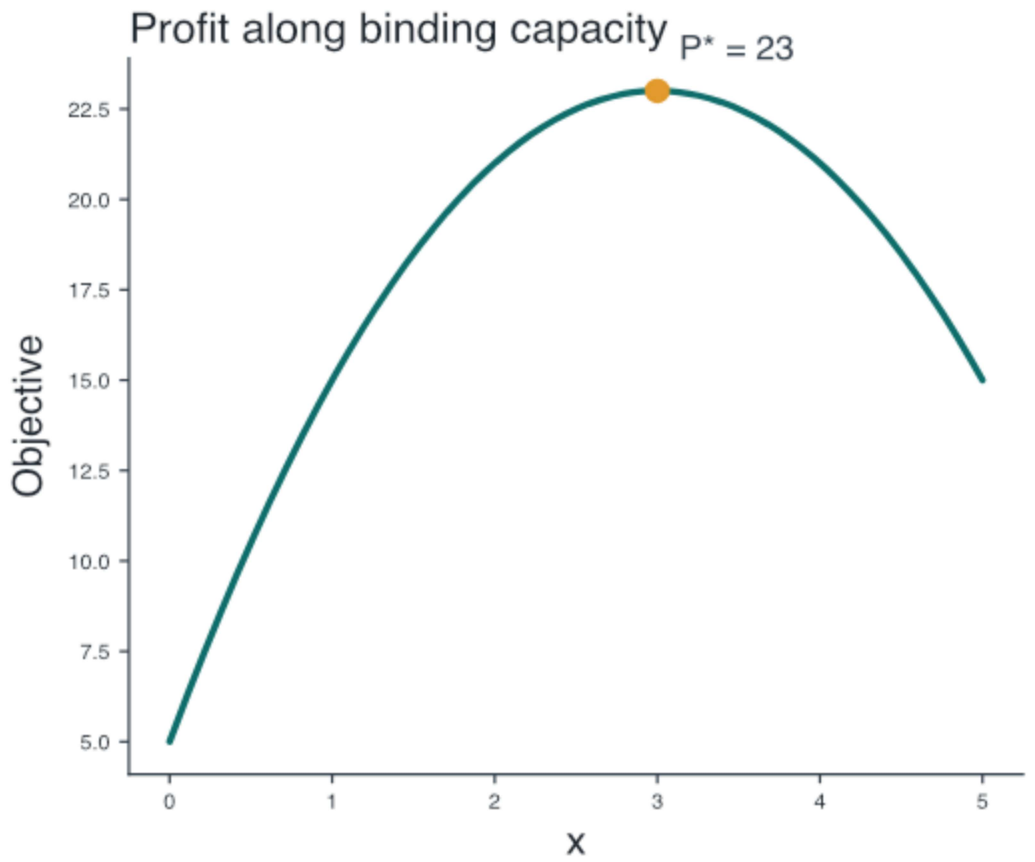
Unconstrained optimum: (4,3), but 4+3 > 5.

Try x,y > 0 with binding capacity → α = β = 0.

$x = 4 - \mu/2,\ \ y = 3 - \mu/2,\ \ x+y = 5$

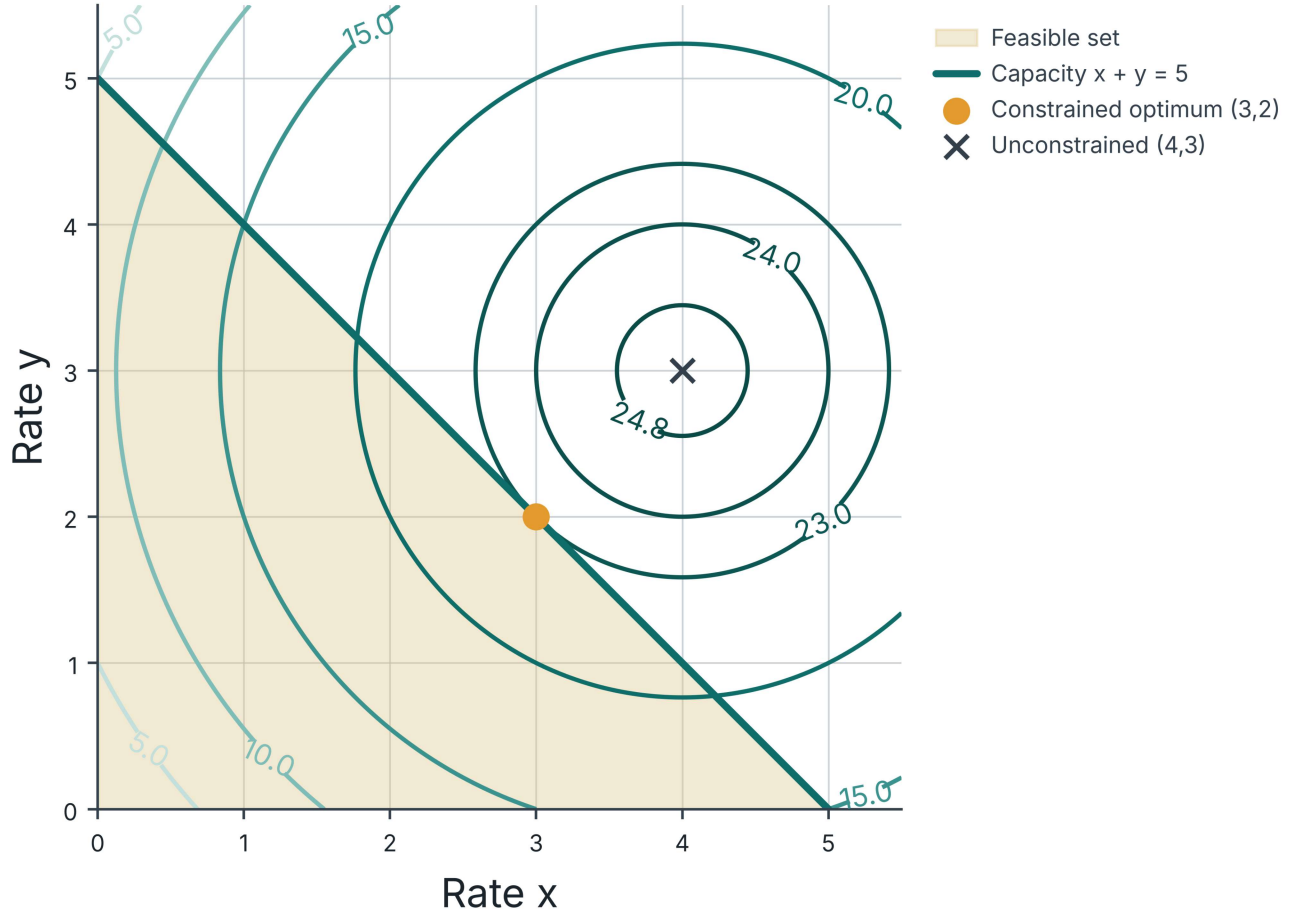
$\mu = 2,\ \ x^* = 3,\ \ y^* = 2,\ \ P^* = 23$

All KKT conditions hold. $H(f) = 2I \succ 0$.



With y = 5−x, profit peaks at x = 3.

Profit contours and the capacity limit



Marginal value of capacity

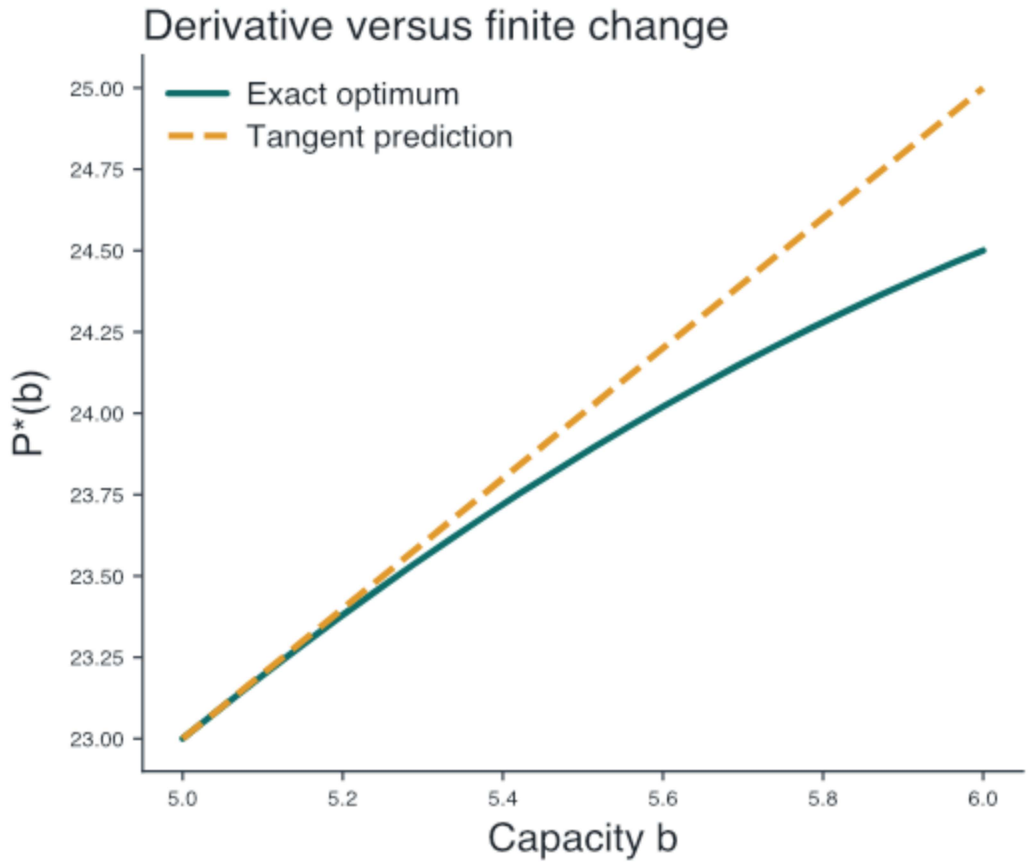
Replace 5 by b. For $1 < b < 7$:

$x^* = (b+1)/2,\ \ y^* = (b-1)/2,\ \ \mu^* = 7-b$

$P^*(b) = 25 - (7-b)^2/2$

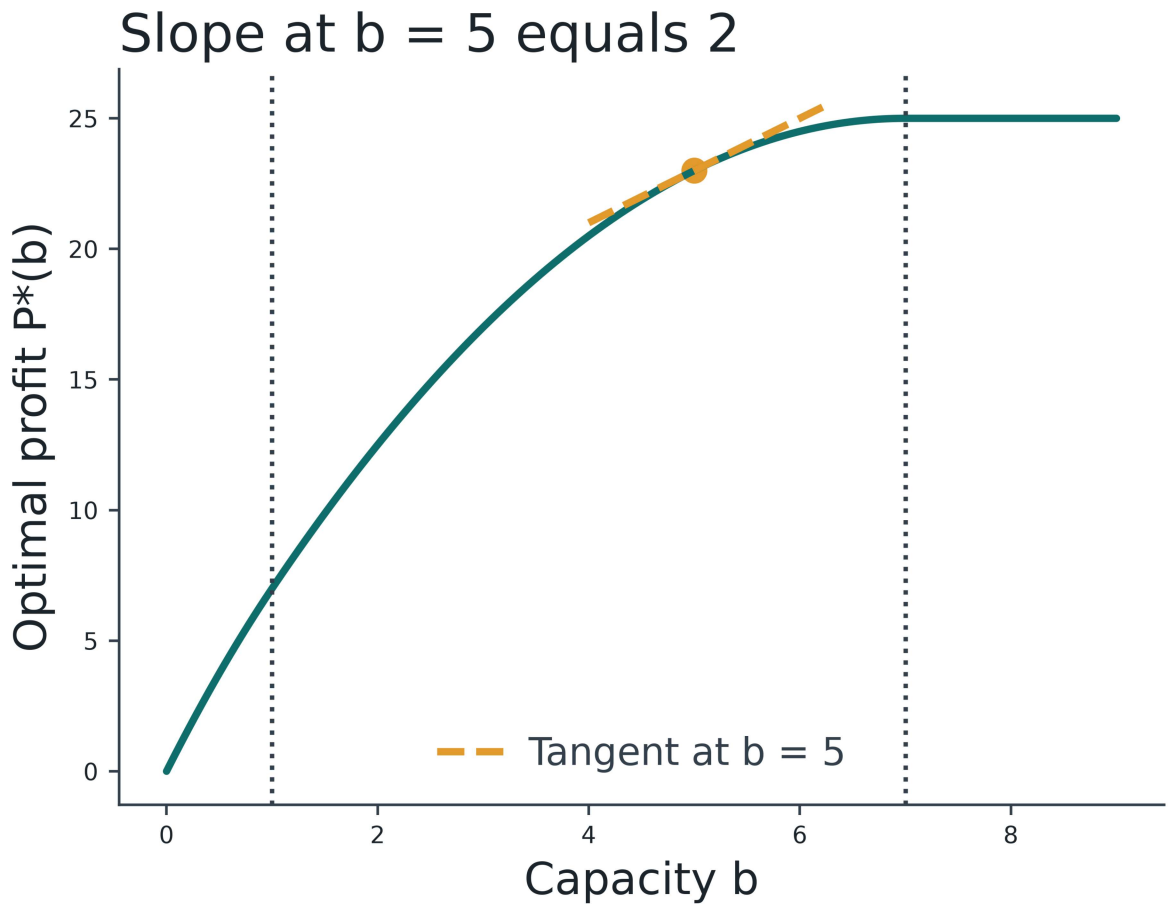
At $b = 5$: $dP^*/db = \mu^* = 2$.

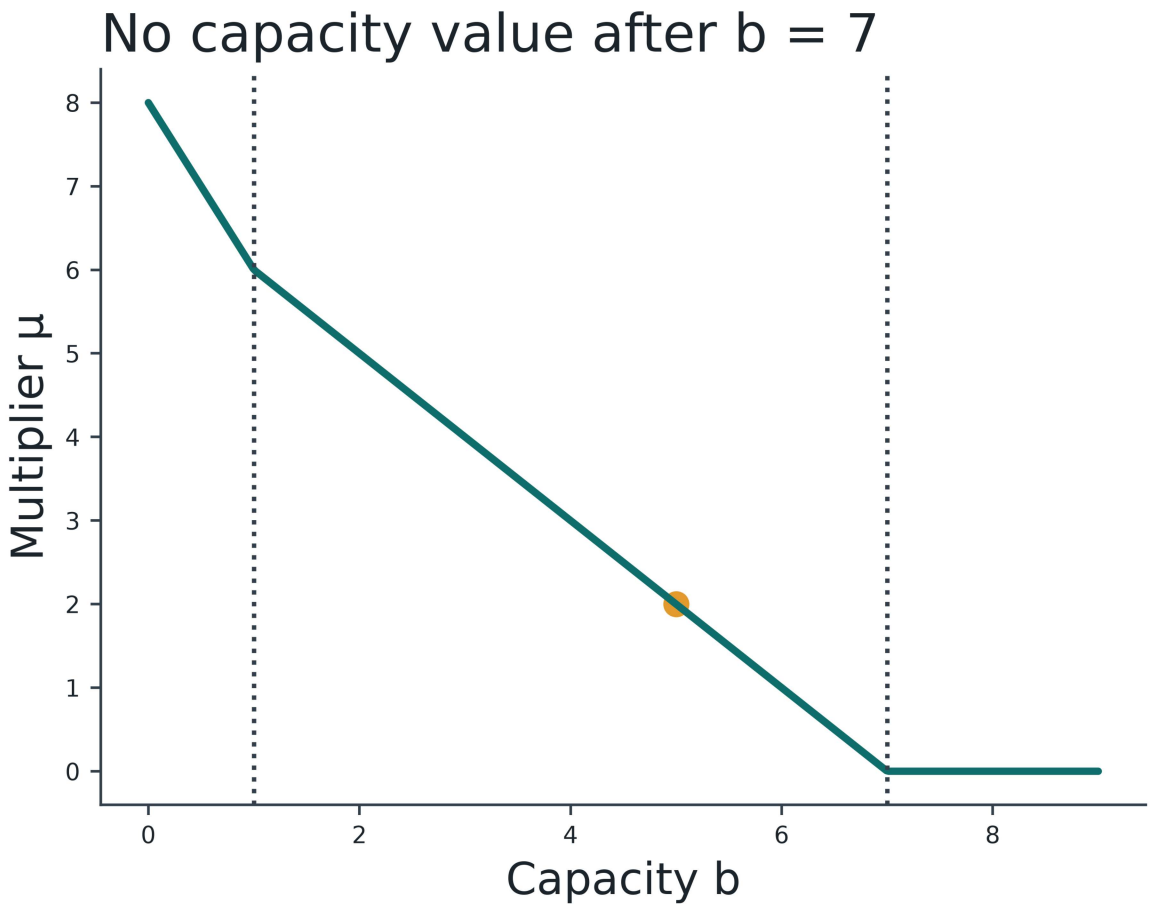
A small increase Δb gives $\Delta P^* \approx 2\Delta b$.



At b = 6, the exact gain is 1.5, not the tangent prediction 2.

Capacity changes the optimum and its marginal value





Maximum and minimum need different certificates

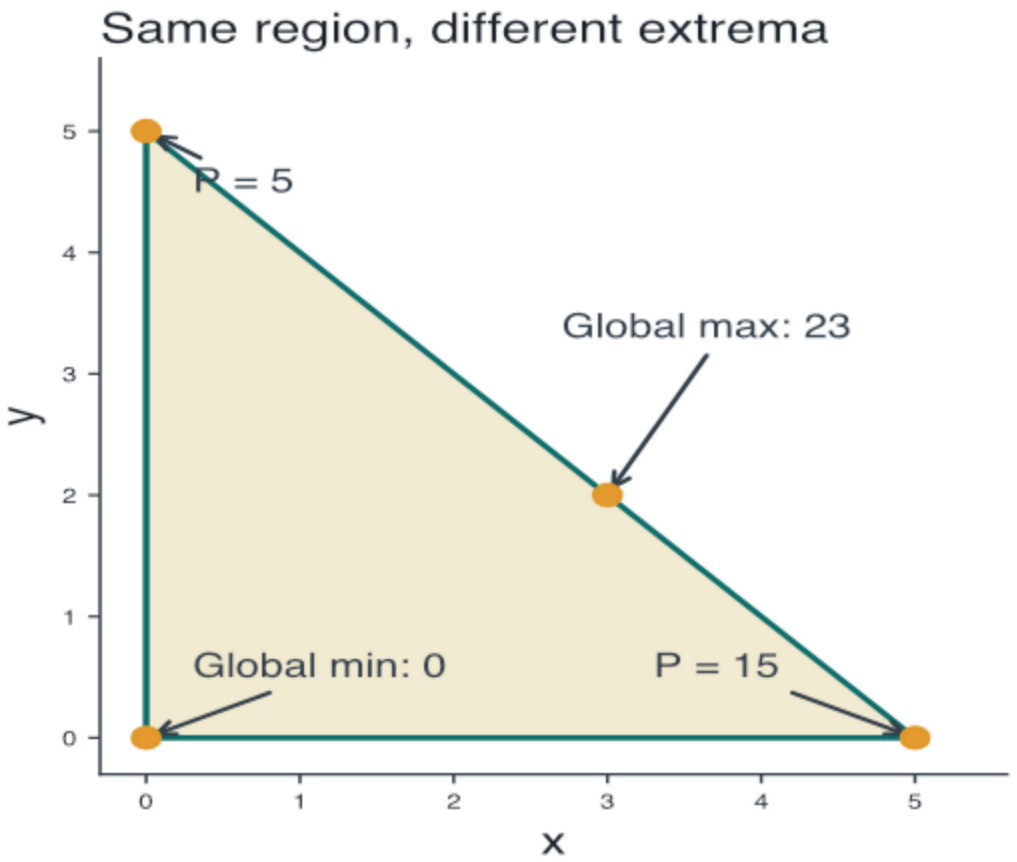
Maximize P → minimize −P, a strictly convex function.

Thus the KKT point (3,2) certifies the unique maximum.

For the minimum of P on the same triangle:

P = x(8−x) + y(6−y) ≥ 0 for 0 ≤ x,y ≤ 5.

P = 0 only at (0,0). Global minimum P* = 0.



The minimum is certified by P ≥ 0, with equality only at the origin.

Practice: constrained quadratic cost

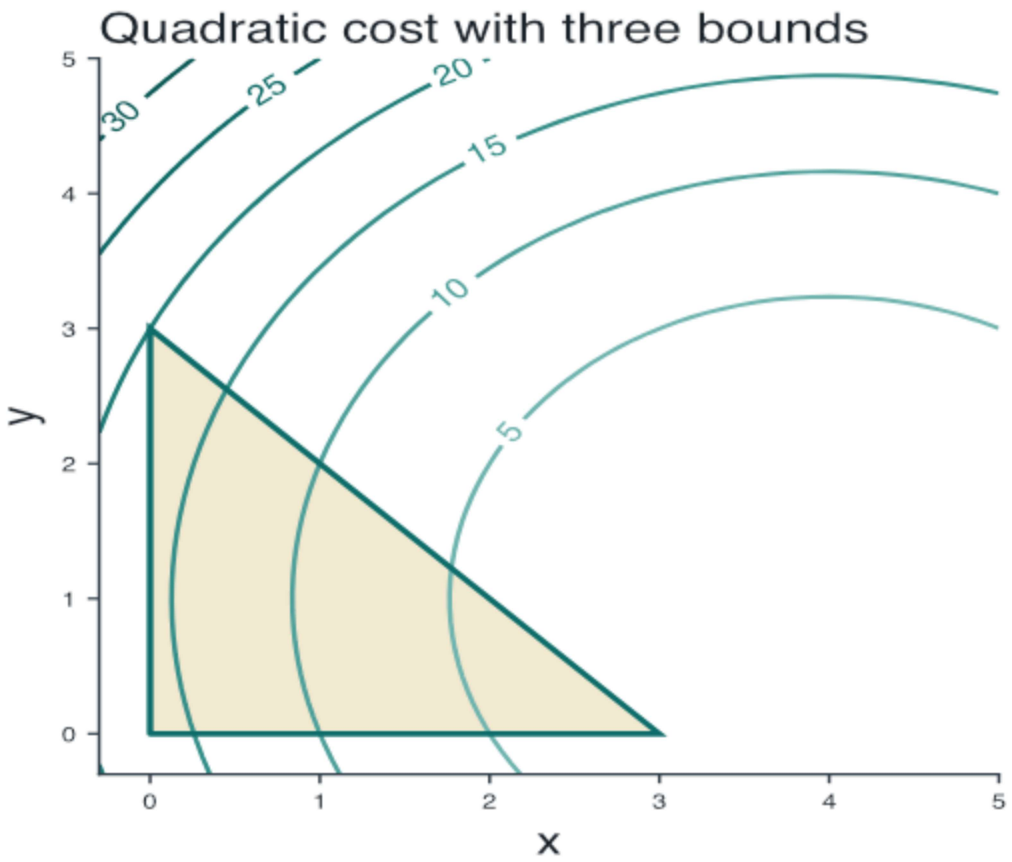
Minimize $C(x,y) = (x-4)^2 + (y-1)^2$

subject to $x+y \le 3$ and $x,y \ge 0$.

Write the full KKT system.

Find x^*, y^*, and all three multipliers.

Explain whether every active constraint has $\mu > 0$.



Inspect the capacity boundary and the nonnegativity bounds.

Practice solution

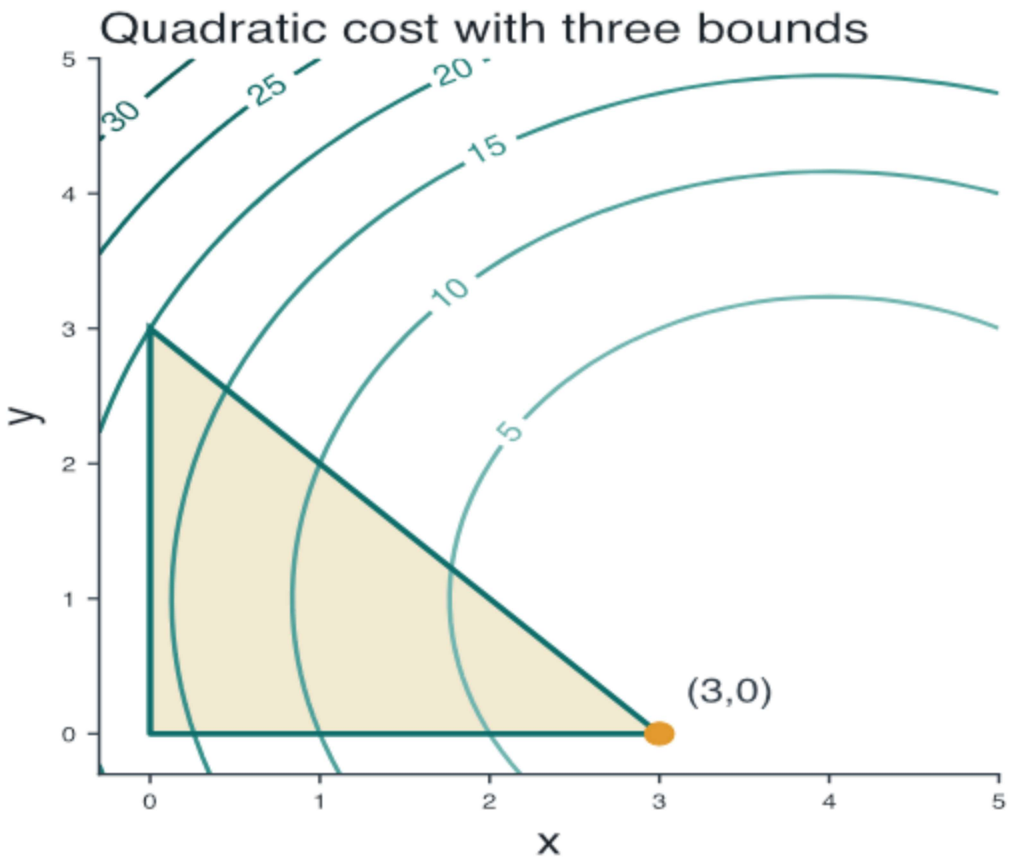
$L = C + \mu(x+y-3) - \alpha x - \beta y$

Stationarity: $2(x-4)+\mu-\alpha = 0$, $2(y-1)+\mu-\beta = 0$

$x^* = 3$, $y^* = 0$, $\mu^* = 2$, $\alpha^* = 0$, $\beta^* = 0$

Capacity and $y \geq 0$ are active, but $\beta^* = 0$.

$C^* = 2$. KKT and strict convexity certify a unique global minimum.



Inspect the capacity boundary and the nonnegativity bounds.

Common mistakes

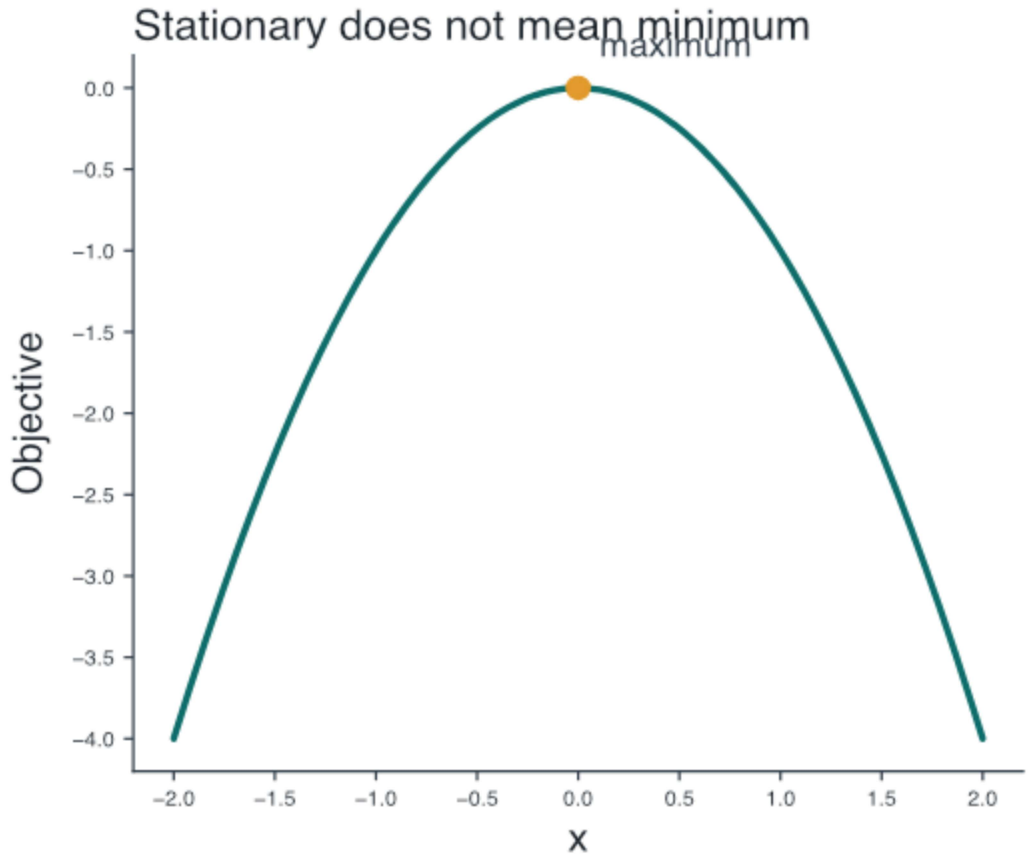
Treating every stationary point as a minimum.

Forgetting endpoints or nonnegativity constraints.

Changing a constraint sign without changing the convention.

Solving stationarity without checking all KKT conditions.

Using KKT alone to claim global optimality in a nonconvex problem.



The origin is stationary, but moving away lowers the objective.

Exit ticket

6 questions · about 6 minutes

Answer individually before leaving.

Hessian eigenvalues, Lagrange geometry,
and the four KKT conditions.

5 scored questions + 1 short explanation.
Submit to see your score and review.

Scan to open the form

No login required. Wait for “Response saved” after submitting.

https://optimization-2105623-class-check.soorathep-k.chatgpt.site/#exit_ticket

Class feedback

5 questions · about 2 minutes

Tell us what was clear and what needs another example.

Comment in Thai or English.

No name or email is requested.

Scan to open the form

No login required. Wait for “Response saved” after submitting.

https://optimization-2105623-class-check.soorathep-k.chatgpt.site/#feedback

Optional: equality second-order test

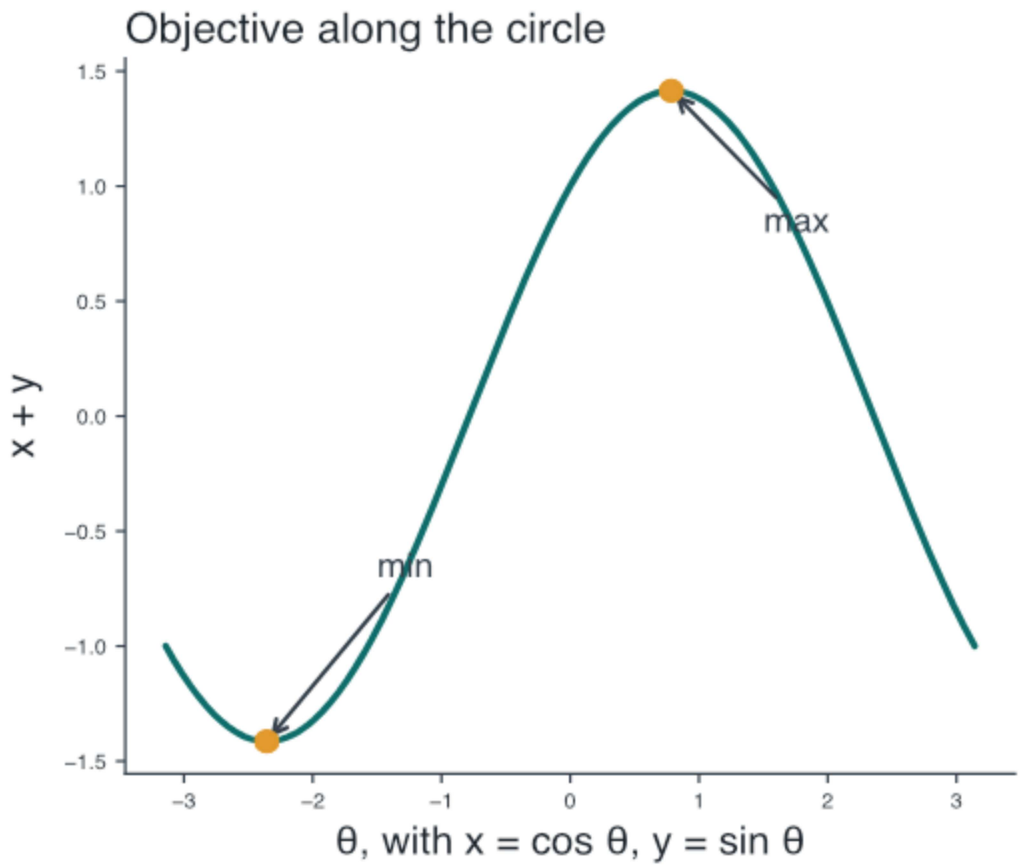
Assume C^2 functions, feasibility, stationarity, and LICQ.

Use $\nabla^2_{xx}L$, not only $\nabla^2 f$.

Test tangent directions d satisfying $J_h d = 0$.

$d^T \nabla^2_{xx} L\, d > 0$ for every nonzero tangent d

is sufficient for a strict local minimum.



Restricted curvature follows feasible motion, not arbitrary directions in space.

References and further practice

Boyd and Vandenberghe, Convex Optimization (2004)

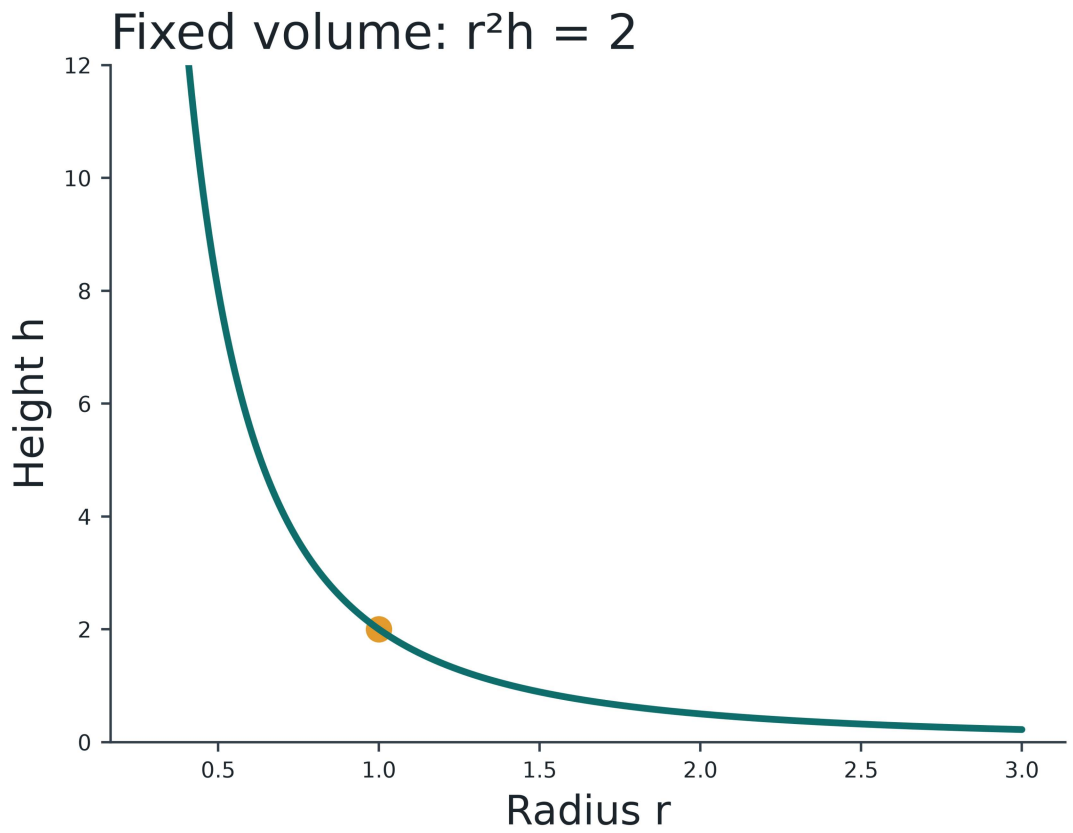
Section 4.2: convex optimization problems

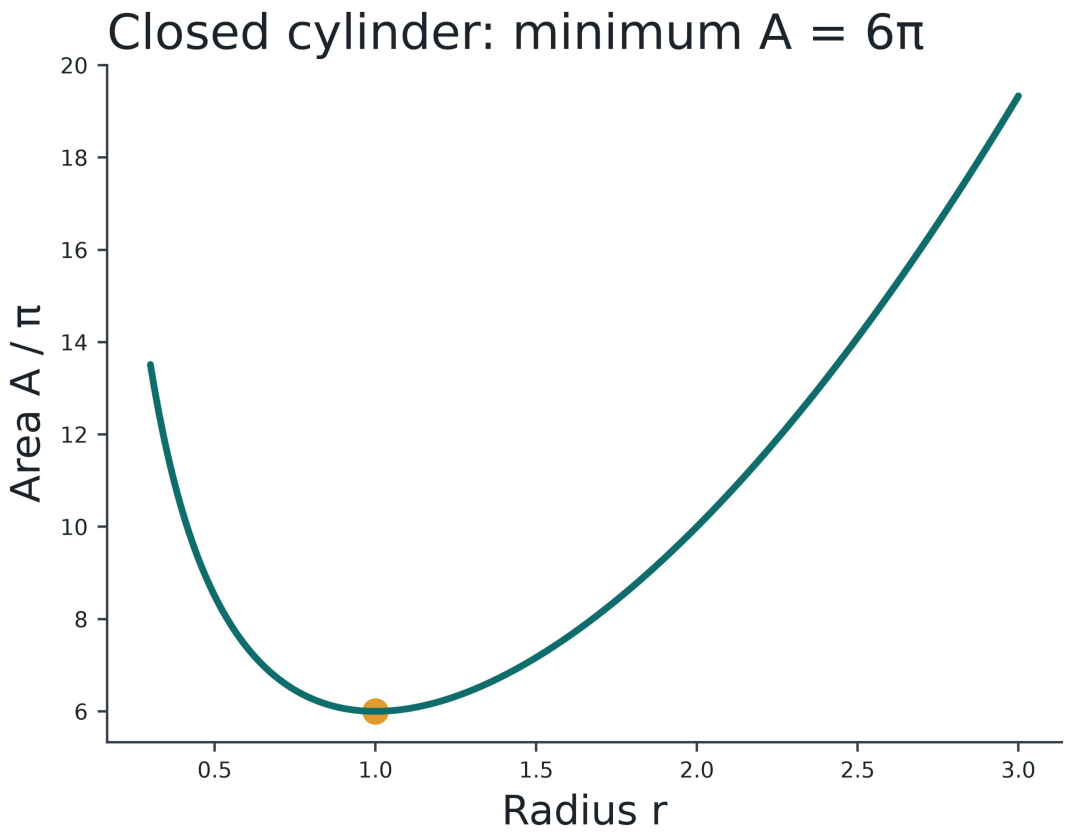
Section 5.5.3: KKT optimality conditions

Section 5.6: perturbation and sensitivity analysis

stanford.edu/~boyd/cvxbook/

Extra Lagrange example: a closed cylindrical tank





Closed tank: minimize $A = 2\pi r^2 + 2\pi rh$, with $\pi r^2 h = 2\pi$ and $r,h > 0$.

Closed cylinder: Lagrange solution

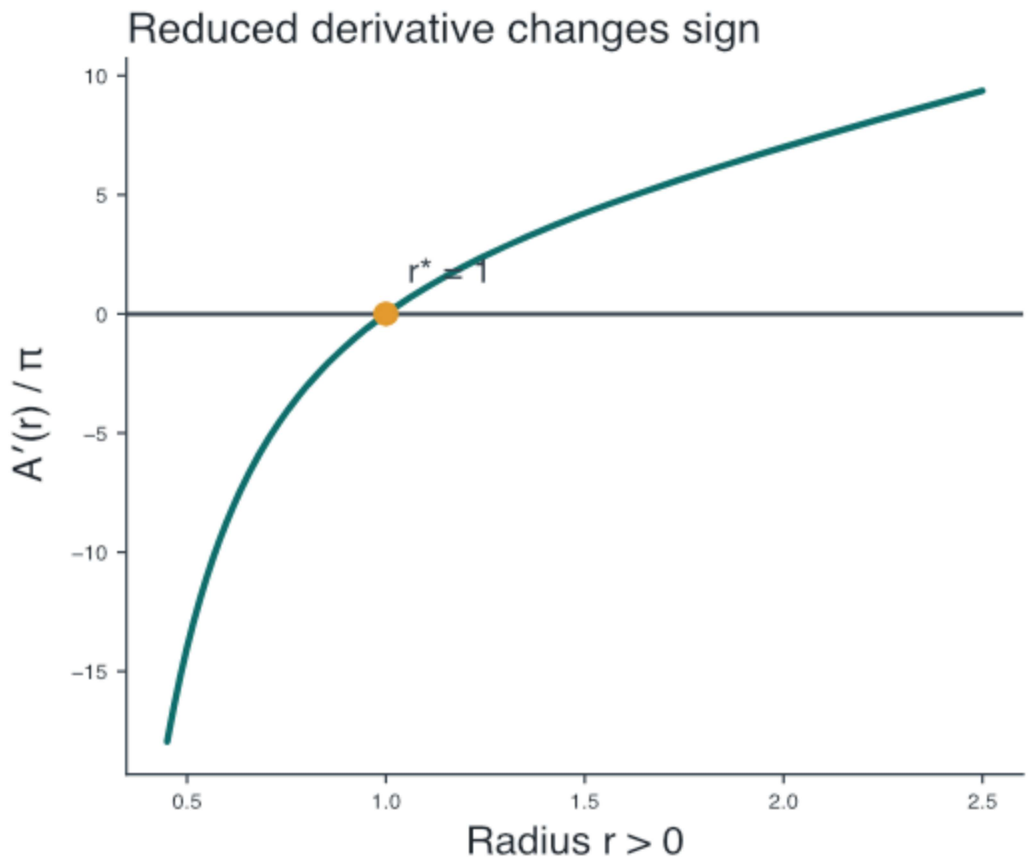
$L = 2\pi r^2 + 2\pi rh + \lambda(\pi r^2 h - 2\pi)$

$\partial L/\partial h = 2\pi r + \lambda \pi r^2 = 0 \rightarrow \lambda = -2/r$

$\partial L/\partial r = 4\pi r + 2\pi h + 2\lambda \pi r h = 0 \rightarrow h = 2r$

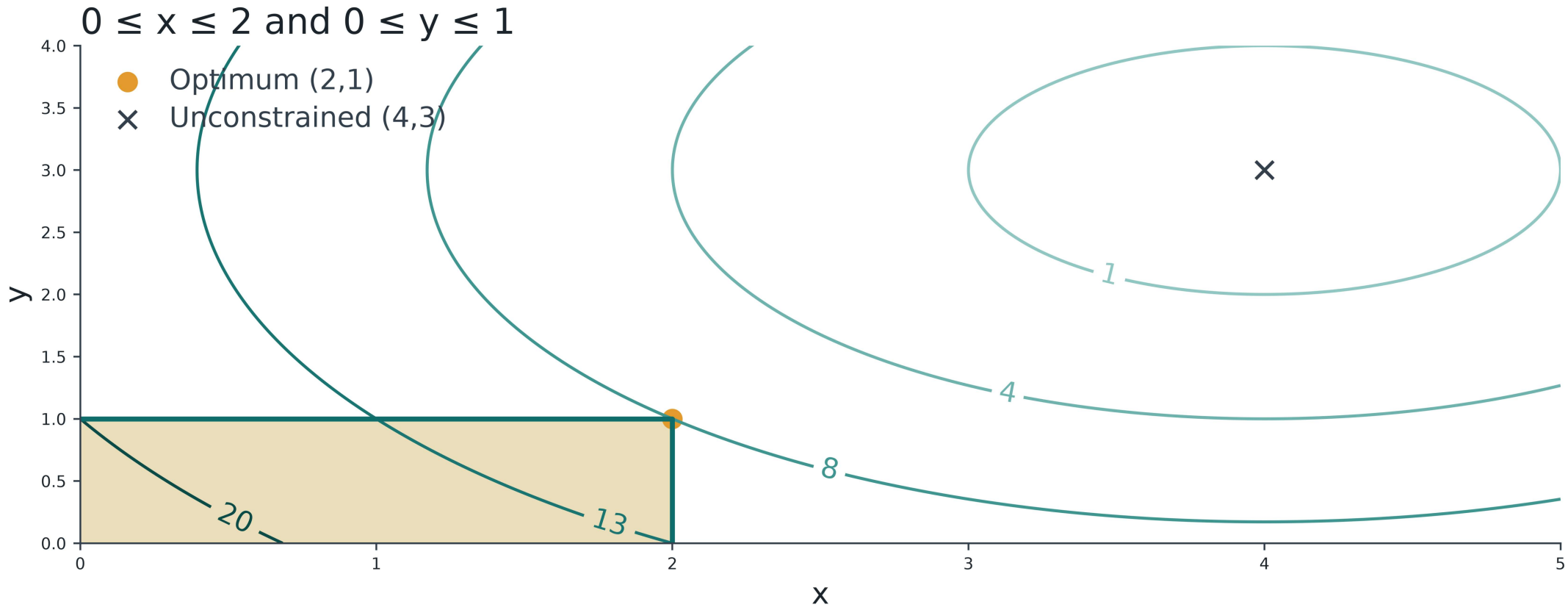
$r^* = 1,\ h^* = 2,\ \lambda^* = -2,\ A^* = 6\pi$

$A(r) = 2\pi r^2 + 4\pi/r$ has $A''(r) > 0$ for $r > 0$.



$A' < 0$ before $r = 1$ and $A' > 0$ after it.

Extra KKT example: two active capacity bounds



Minimize $f = (x-4)^2 + (y-3)^2$, with $0 \le x \le 2$ and $0 \le y \le 1$.

Two active bounds: KKT solution

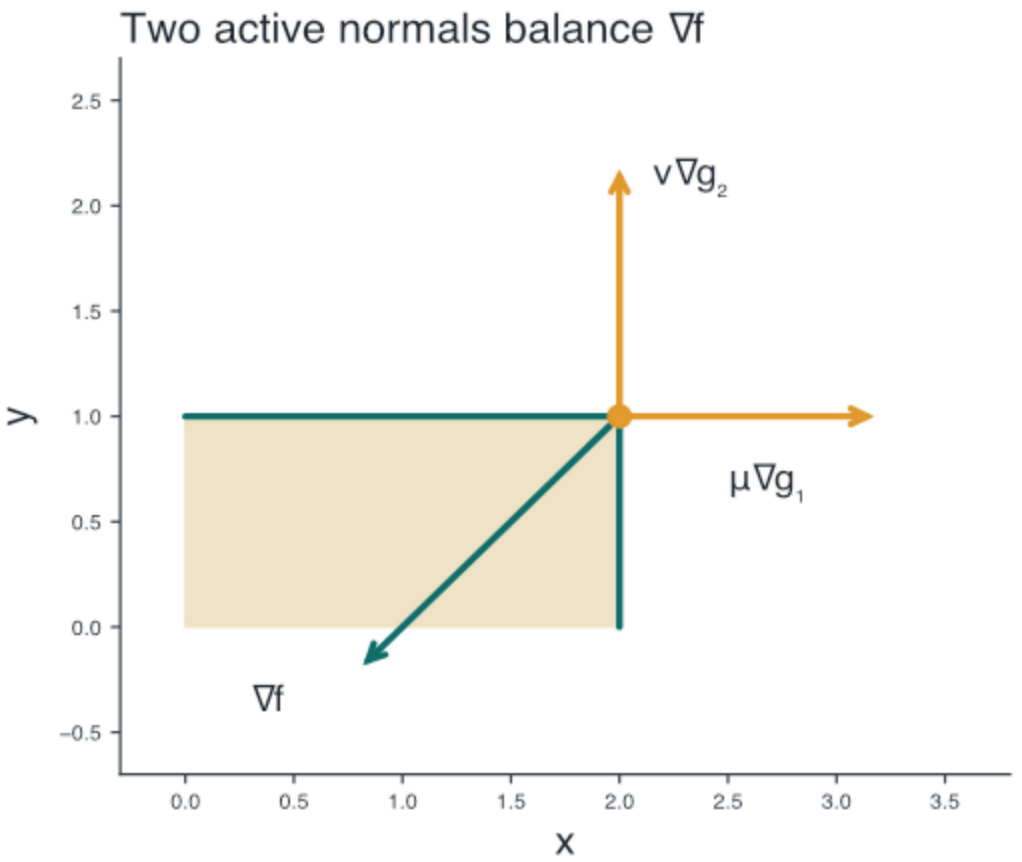
$L = (x-4)^2+(y-3)^2 + \mu(x-2)+\nu(y-1) - \alpha x-\beta y$

$2(x-4)+\mu-\alpha = 0,\ \ 2(y-3)+\nu-\beta = 0$

At (2,1): $\mu = 4$, $\nu = 4$, $\alpha = \beta = 0$, $f^* = 8$

Constraint values: (0,0,−2,−1). All products are zero.

$H(f) = 2I$ and affine constraints → unique global minimum.



Arrows share one scale: (4,0) + (0,4) balances (−4,−4).