LP and duality: questions explained

Where do the dual variables come from?

When does a shadow price stop working?

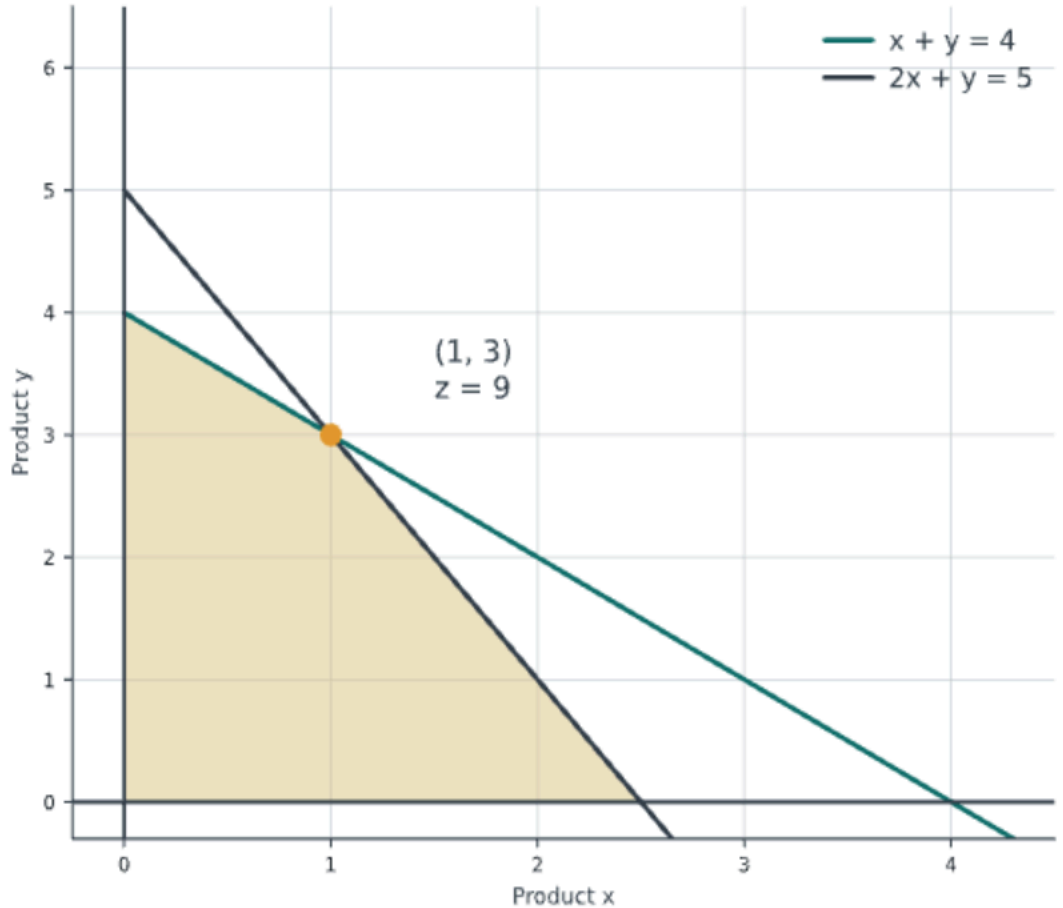
Worked example: one extra resource, a different optimum.

The original production problem

Maximize $z = 3x + 2y$

Resource 1: $x + y \leq 4$
Resource 2: $2x + y \leq 5$

$x \geq 0, y \geq 0$



Production quantities and resource prices

The primal chooses product quantities.
The dual chooses resource prices.

z and w denote objective values.
They are not additional independent decisions.

Symbol	Meaning
x, y	Product quantities
u, v	Prices per resource unit
z = 3x + 2y	Production profit
w = 4u + 5v	Resource value

Step 1: weighting the resource constraints

$u(x + y) \leq 4u$ when $u \geq 0$

$v(2x + y) \leq 5v$ when $v \geq 0$

Adding gives:
$u(x + y) + v(2x + y) \leq 4u + 5v$

Step 2: collecting product coefficients

$ux + uy + 2vx + vy$

$= (u + 2v)x + (u + v)y$

Cost assigned to one x: u + 2v
Cost assigned to one y: u + v

Per product unit	x	y
Resource 1 at price u	1u	1u
Resource 2 at price v	2v	1v
Total assigned cost	u + 2v	u + v

Step 3: a valid upper bound on profit

$u + 2v \geq 3$ covers the profit of x

$u + v \geq 2$ covers the profit of y

$$3x + 2y \leq (u + 2v)x + (u + v)y$$
$$\leq 4u + 5v$$

Step 4: the complete dual problem

Minimize $w = 4u + 5v$

$u + 2v \geq 3$
$u + v \geq 2$
$u \geq 0, v \geq 0$

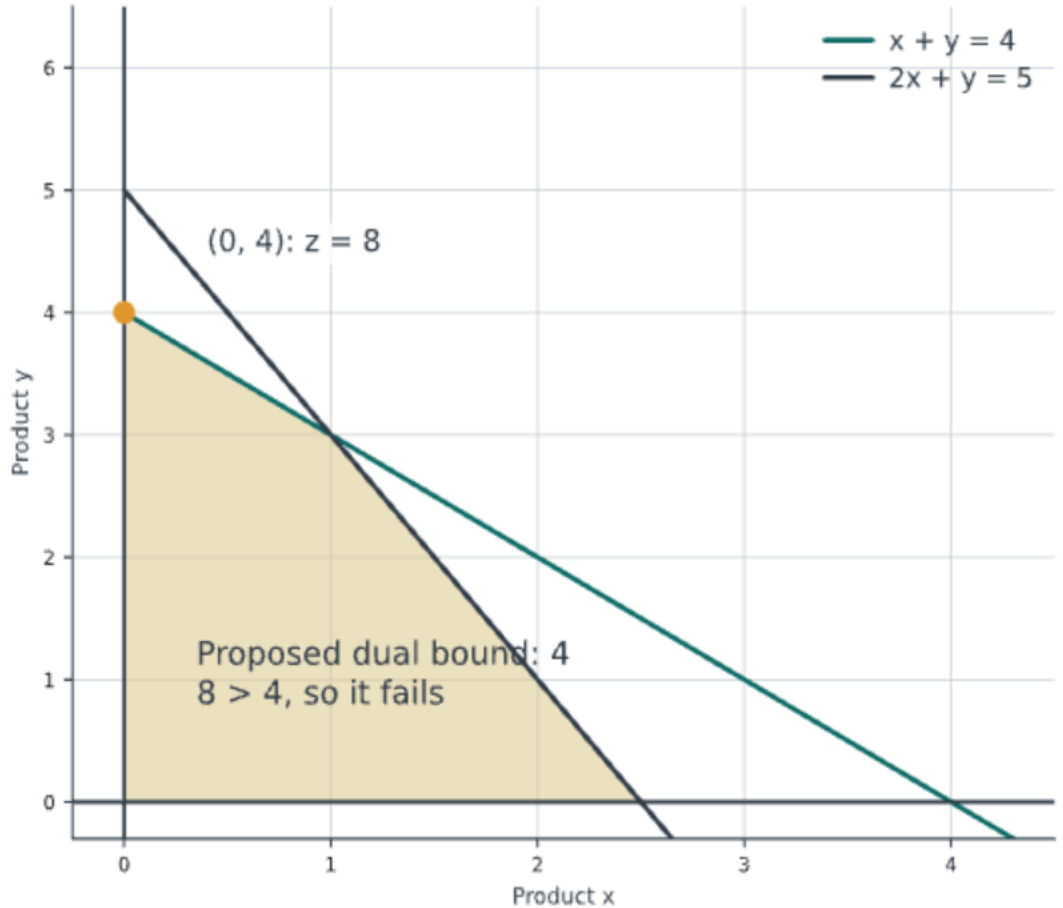
Choose the smallest valid resource-value bound.

A cheap price pair can fail the bound

Try u = 1, v = 0: w = 4

u + 2v = 1 < 3
u + v = 1 < 2

Feasible production (x, y) = (0, 4)
has z = 8 > 4. The proposed bound fails.

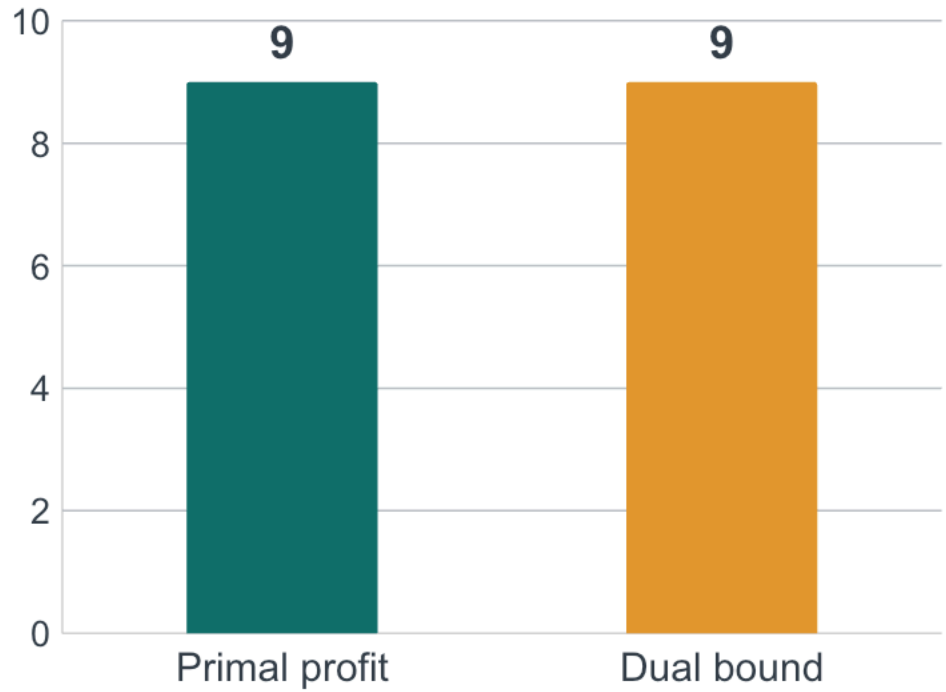


Matching feasible values certify optimality

Primal: (x, y) = (1, 3)
x + y = 4, 2x + y = 5, z = 9

Dual: (u, v) = (1, 1)
u + 2v = 3, u + v = 2, w = 9

Feasible profit = feasible upper bound = 9



Feasible values match at 9

Complementary slackness links the solutions

$u(4 - x - y) = 0$
$v(5 - 2x - y) = 0$

$x(u + 2v - 3) = 0$
$y(u + v - 2) = 0$

At (1, 3), both products are positive.
Thus $u + 2v = 3$ and $u + v = 2$.

Changing the amount of resource 1

Maximize $3x + 2y$
$x + y \le b_1, \quad 2x + y \le 5$

Baseline: $b_1 = 4$, optimum $z^* = 9$

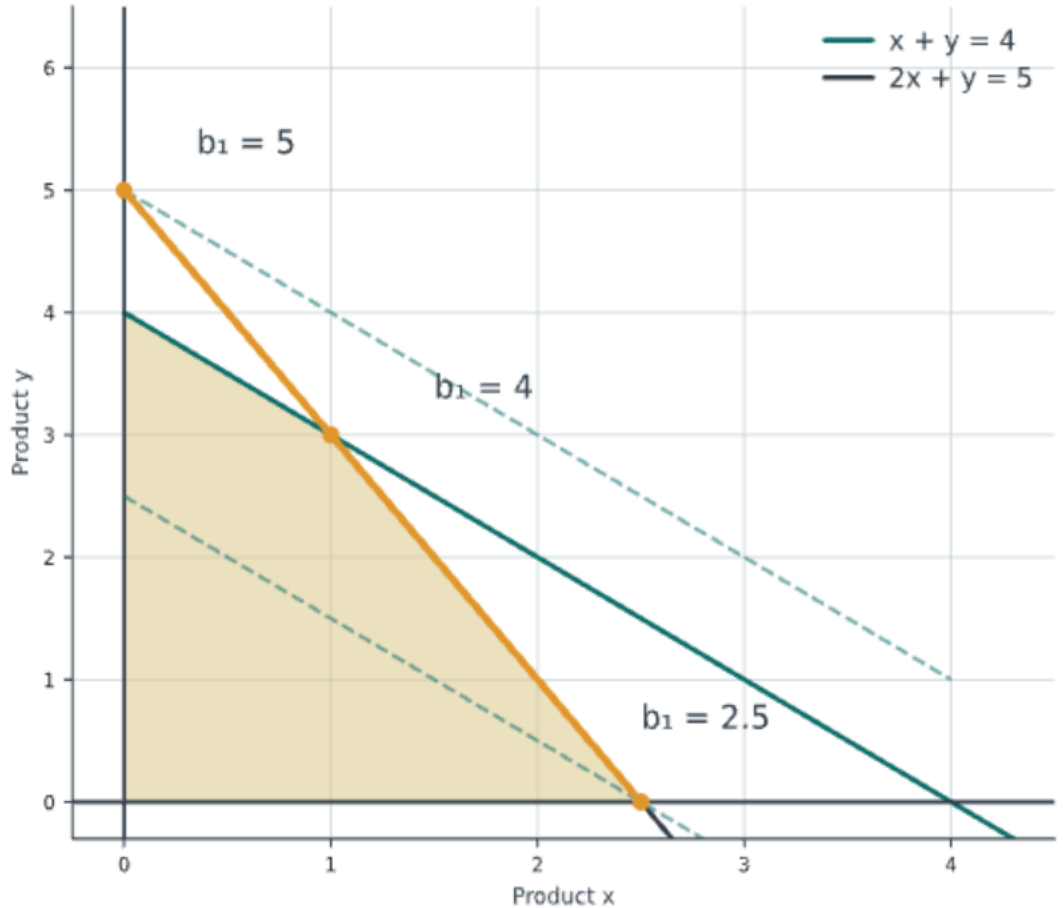
Candidate shadow-price prediction:
$z^*(b_1) = 9 + 1(b_1 - 4)$

Step 1: the intersection as b_1 changes

$x + y = b_1$
$2x + y = 5$

Subtract the first equation from the second:
$x = 5 - b_1$

Then $y = b_1 - x = 2b_1 - 5$

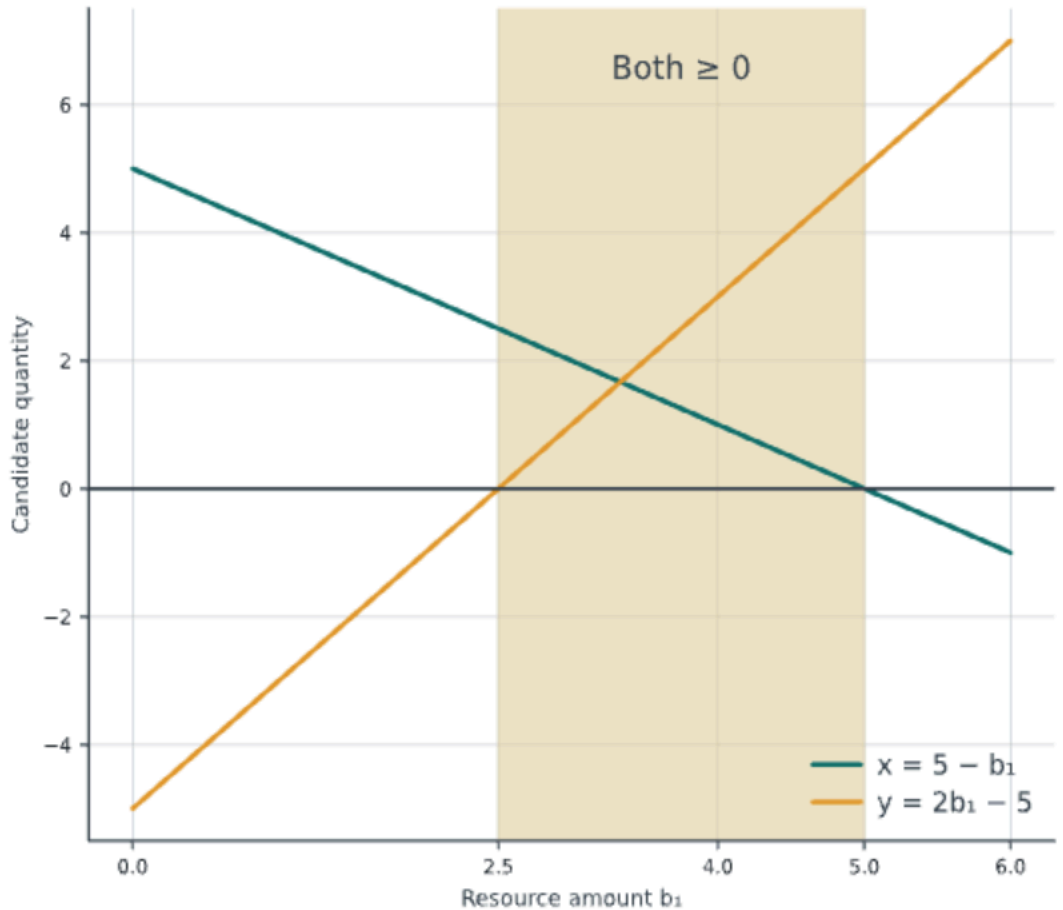


Step 2: the allowable interval

$x = 5 - b_1 \geq 0$ gives $b_1 \leq 5$

$y = 2b_1 - 5 \geq 0$ gives $b_1 \geq 2.5$

Therefore: $2.5 \leq b_1 \leq 5$
This is an interval for b_1, not for x.

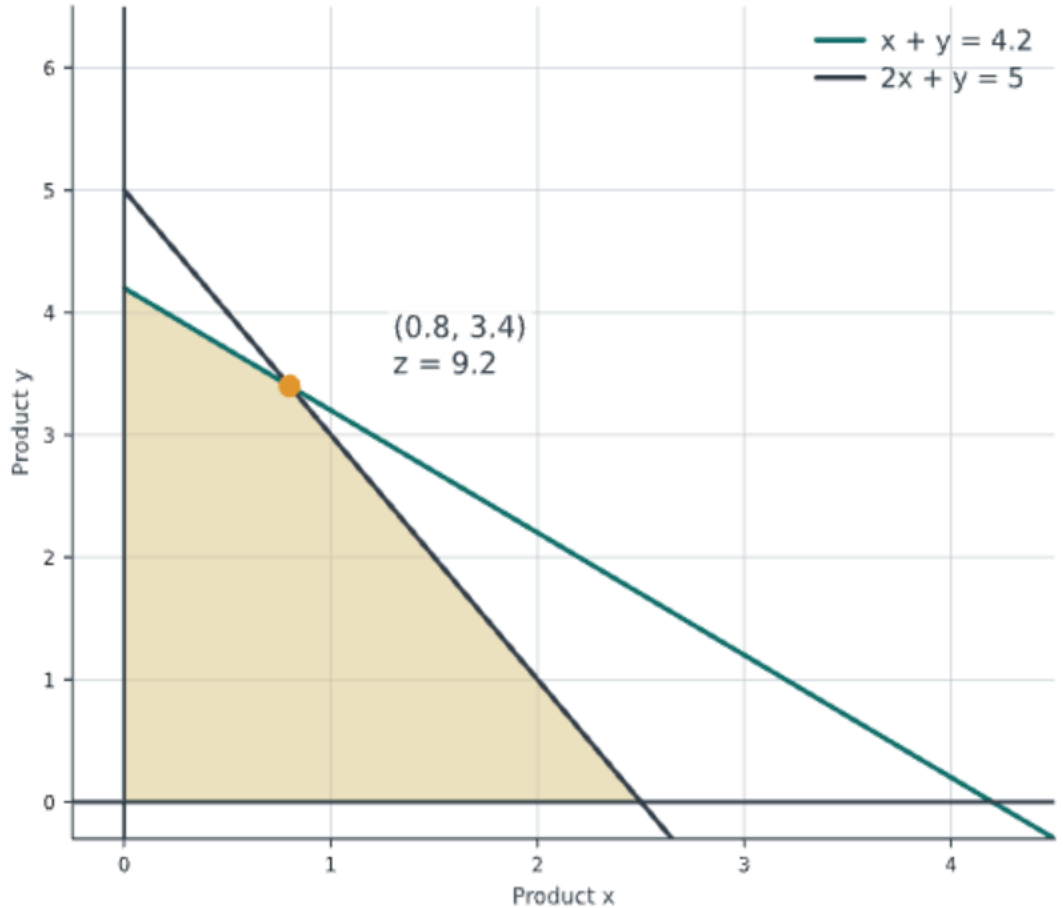


Step 3: profit inside the interval

$z^* = 3(5 - b_1) + 2(2b_1 - 5)$

$= 15 - 3b_1 + 4b_1 - 10$
$= b_1 + 5$

At $b_1 = 4.2$: $x = 0.8$, $y = 3.4$
$z^* = 9.2 = 9 + 1(0.2)$



Beyond the interval: $b_1 = 6$

Old intersection: $x = 5 - 6 = -1$
$y = 2(6) - 5 = 7$

$x = -1$ violates $x \geq 0$.

Old-price prediction: $9 + 1(6 - 4) = 11$
We must solve the changed problem.

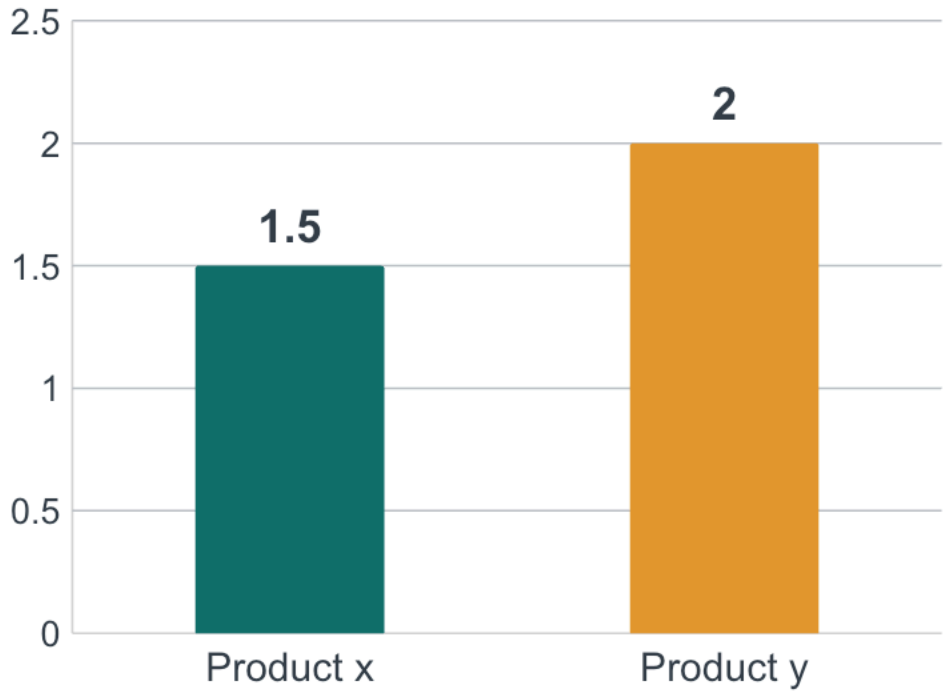


Profit per unit of the limiting resource

At $b_1 = 6$, resource 2 limits production.

Product x: profit / resource 2 = 3 / 2 = 1.5

Product y: profit / resource 2 = 2 / 1 = 2
Here, y earns more per limiting resource unit.



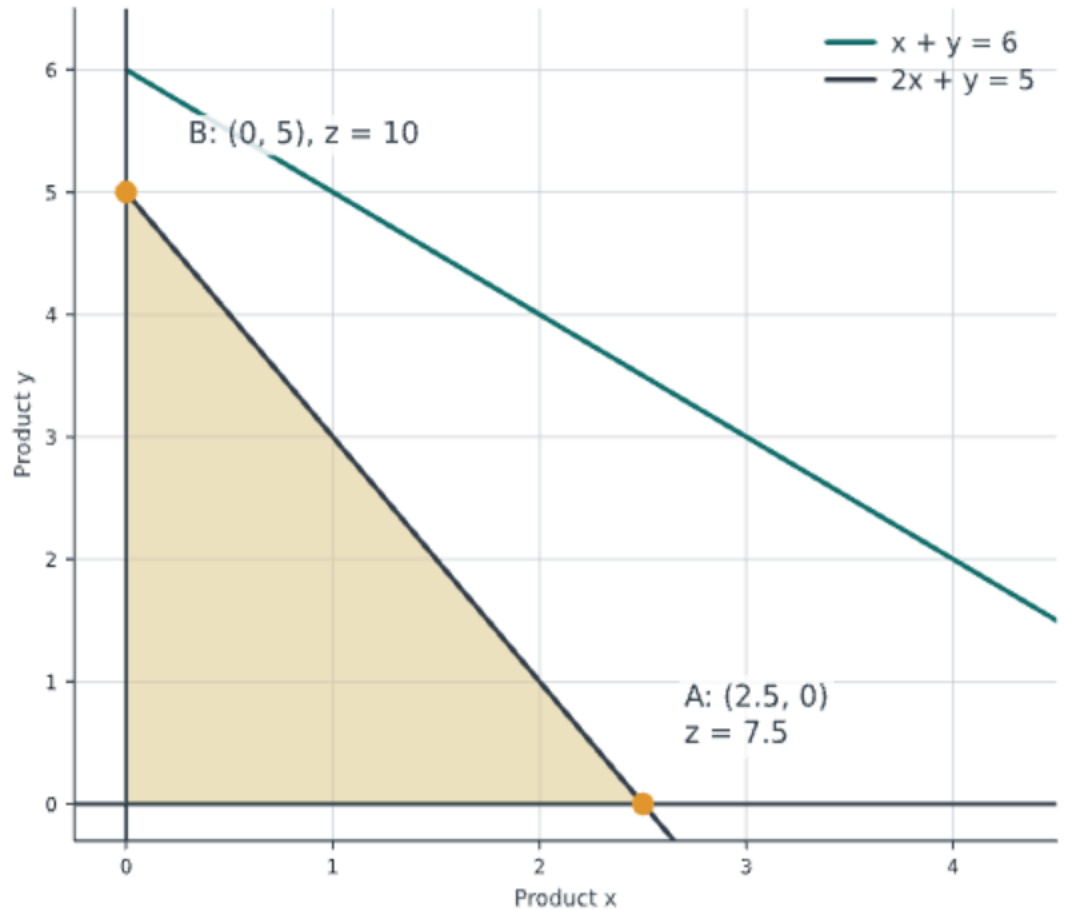
Profit per unit of resource 2

The new optimum uses five units of y

Candidate A: (2.5, 0), profit = 7.5

Candidate B: (0, 5), profit = 10
Resource use: 5 ≤ 6 and 5 ≤ 5

On 2x + y = 5:
y = 5 − 2x, so z = 10 − x.

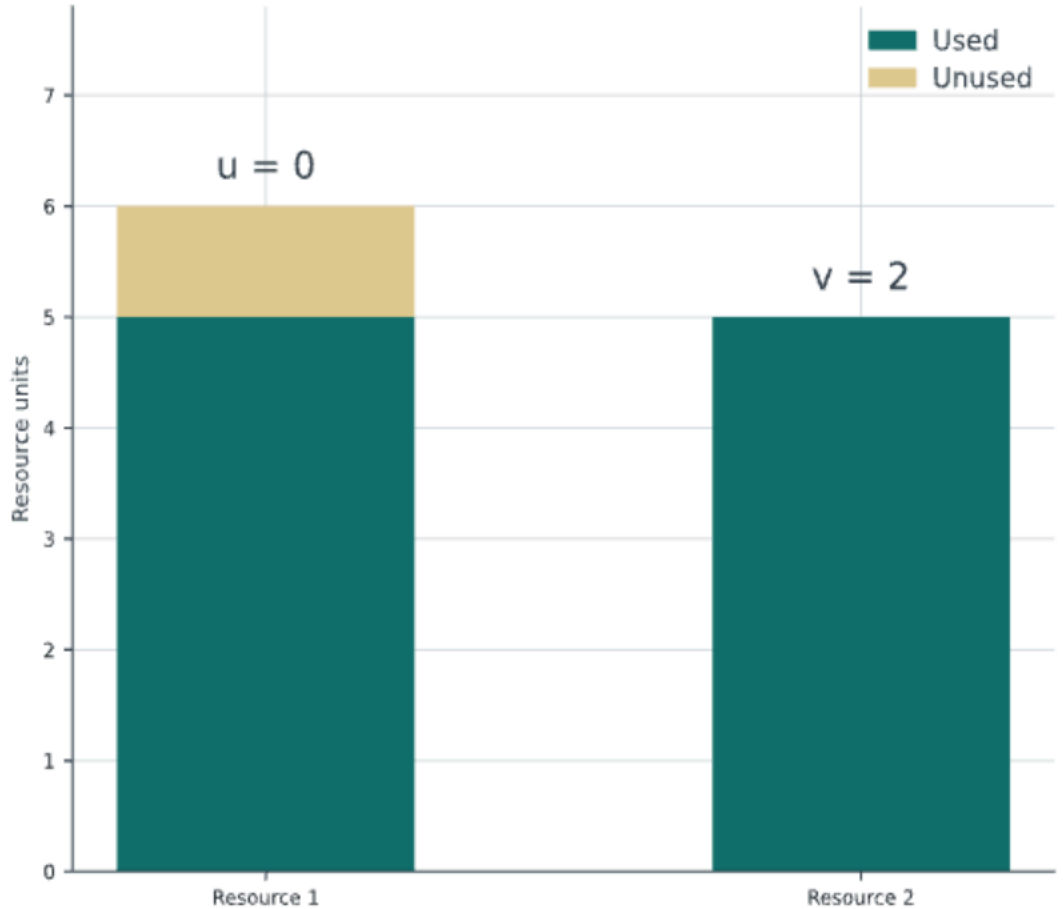


A new dual certificate proves the answer

Choose u = 0, v = 2.
u + 2v = 4 ≥ 3, u + v = 2 ≥ 2

w = 6(0) + 5(2) = 10 = z*

Resource 1 has slack: 6 − 5 = 1.
Its new shadow price is zero.



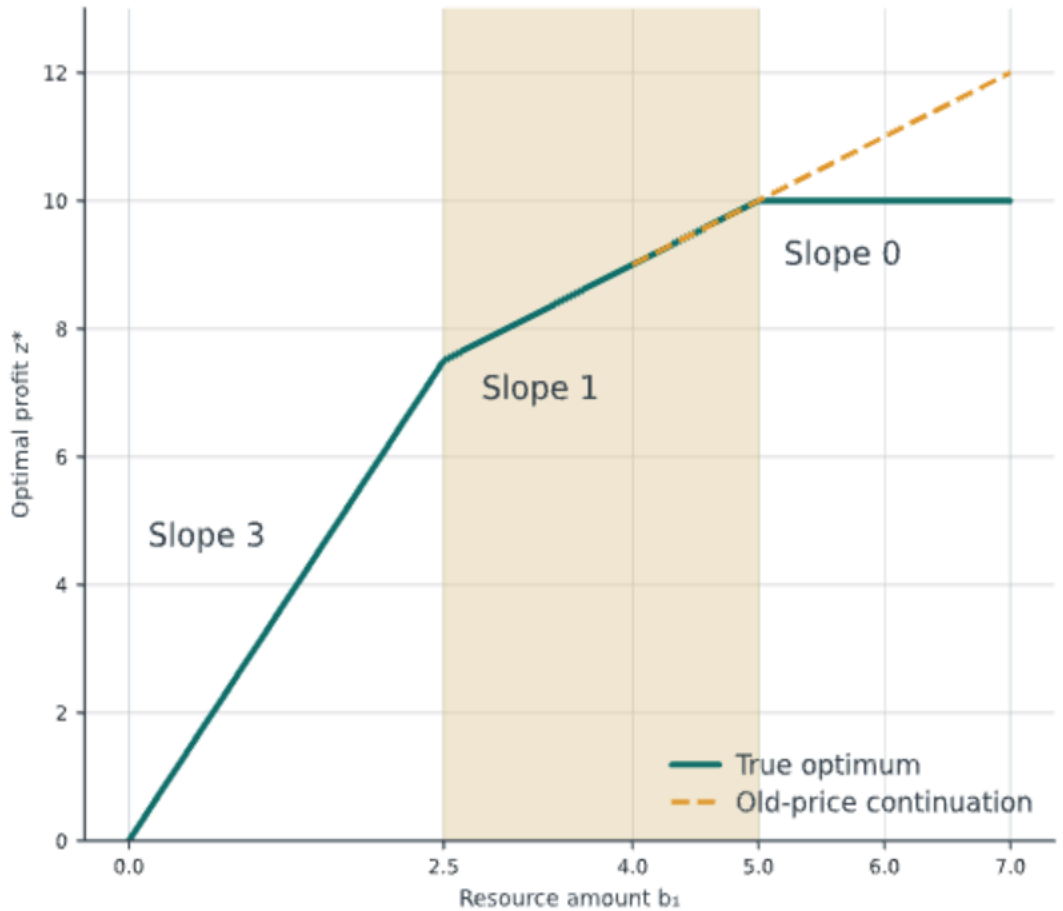
The complete value curve for $b_1 \geq 0$

$0 \leq b_1 \leq 2.5$: $z^* = 3b_1$

$2.5 \leq b_1 \leq 5$: $z^* = b_1 + 5$

$b_1 \geq 5$: $z^* = 10$

Slopes inside the regions: 3, 1, 0

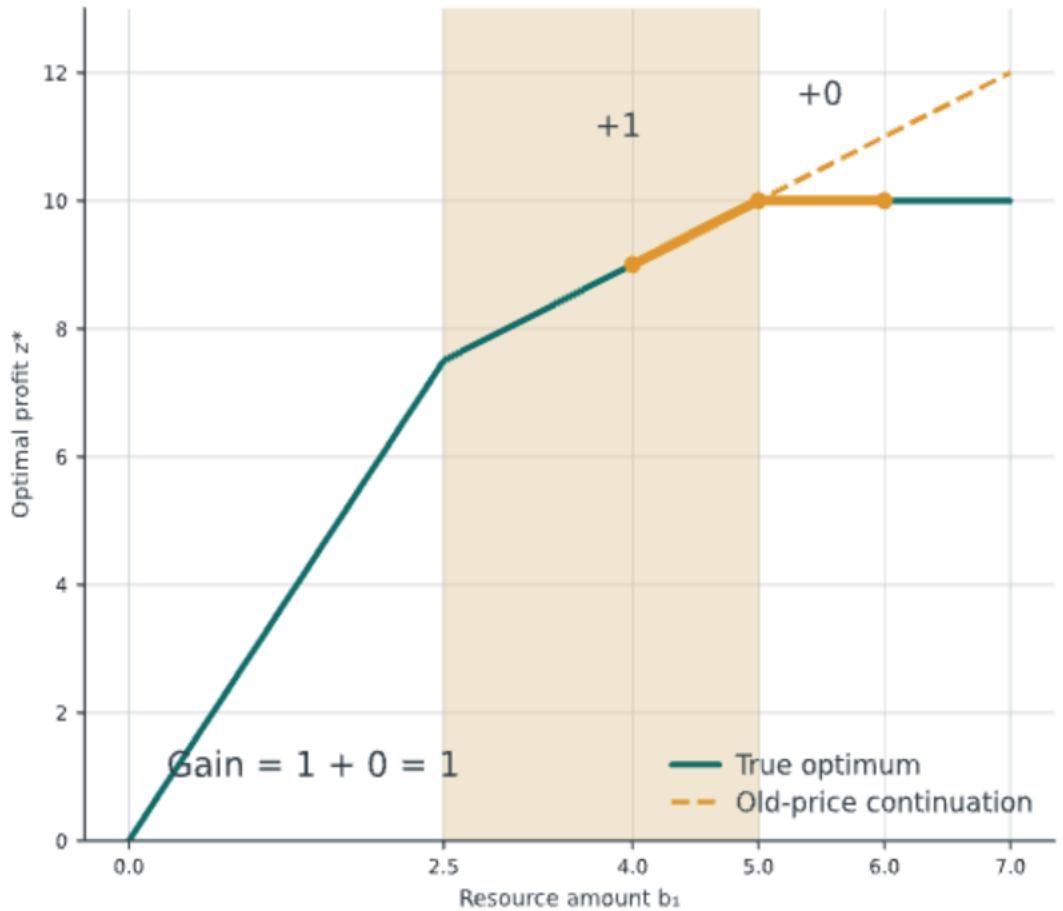


Range endpoints and a finite increase

From $b_1 = 4$ to 5: gain = $1 \times 1 = 1$

From $b_1 = 5$ to 6: gain = $0 \times 1 = 0$

Total gain: 1, so profit rises from 9 to 10.
At $b_1 = 5$, left slope = 1, right slope = 0.

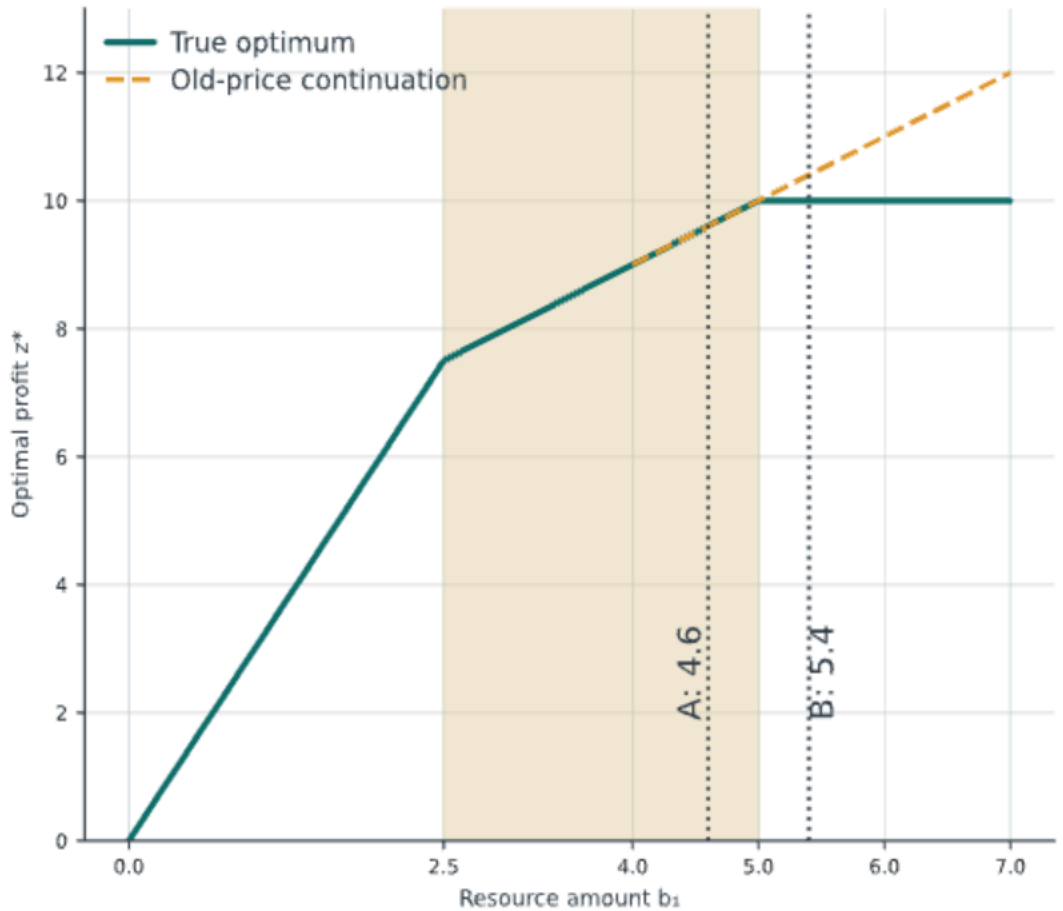


Pause and try the calculation

Keep $2x + y \le 5$ and profit $3x + 2y$.

A. At $b_1 = 4.6$, find x, y, and z*.
B. At $b_1 = 5.4$, find x, y, and z*.

For each case, give feasible dual prices.
Explain why the two shadow prices differ.



Practice solution and optimality checks

A. $b_1 = 4.6$: $x = 0.4$, $y = 4.2$
$z^* = 1.2 + 8.4 = 9.6$
$(u, v) = (1, 1)$: $w = 4.6 + 5 = 9.6$

B. $b_1 = 5.4$: $x = 0$, $y = 5$
$z^* = 10$
$(u, v) = (0, 2)$: $w = 0 + 10 = 10$

A uses both resources fully.
B leaves 0.4 unit of resource 1 unused.

A connected view of primal and dual

Primal: feasible product quantities and profit.

Dual: feasible resource prices and an upper bound.

Matching values certify optimality.
Complementary slackness explains the match.
Shadow prices describe value changes within a regime.