

Integer and mixed-integer programming

2105623 · November 2, 2026

Choosing machines, opening production lines, and verifying decisions

Your three-hour study route

15 min review + 30 min integer geometry

35 min modeling + 10 min break

25 min search + 45 min lab + 20 min Exit Ticket

Recall the LP certificate

A feasible plan gives a candidate value.

A feasible dual solution bounds all primal values.

Matching values certify LP optimality.

Three kinds of decisions

Continuous: tonnes produced, $q \geq 0$.

Integer: number of machines, $n \in \{0, 1, 2, \dots\}$.

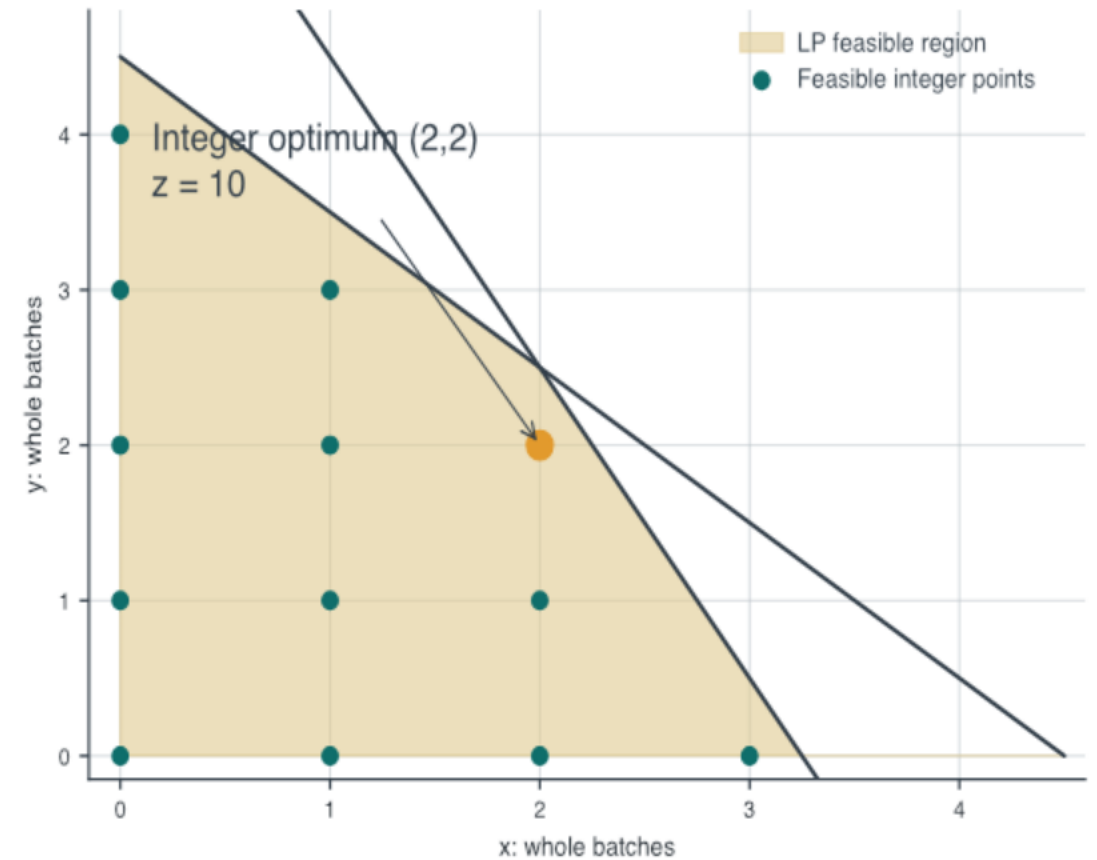
Binary: open a line, $y \in \{0, 1\}$.

A small integer production problem

Maximize $3x + 2y$.

$x + y \leq 4.5$; $2x + y \leq 6.5$.

x and y count batches: nonnegative integers.

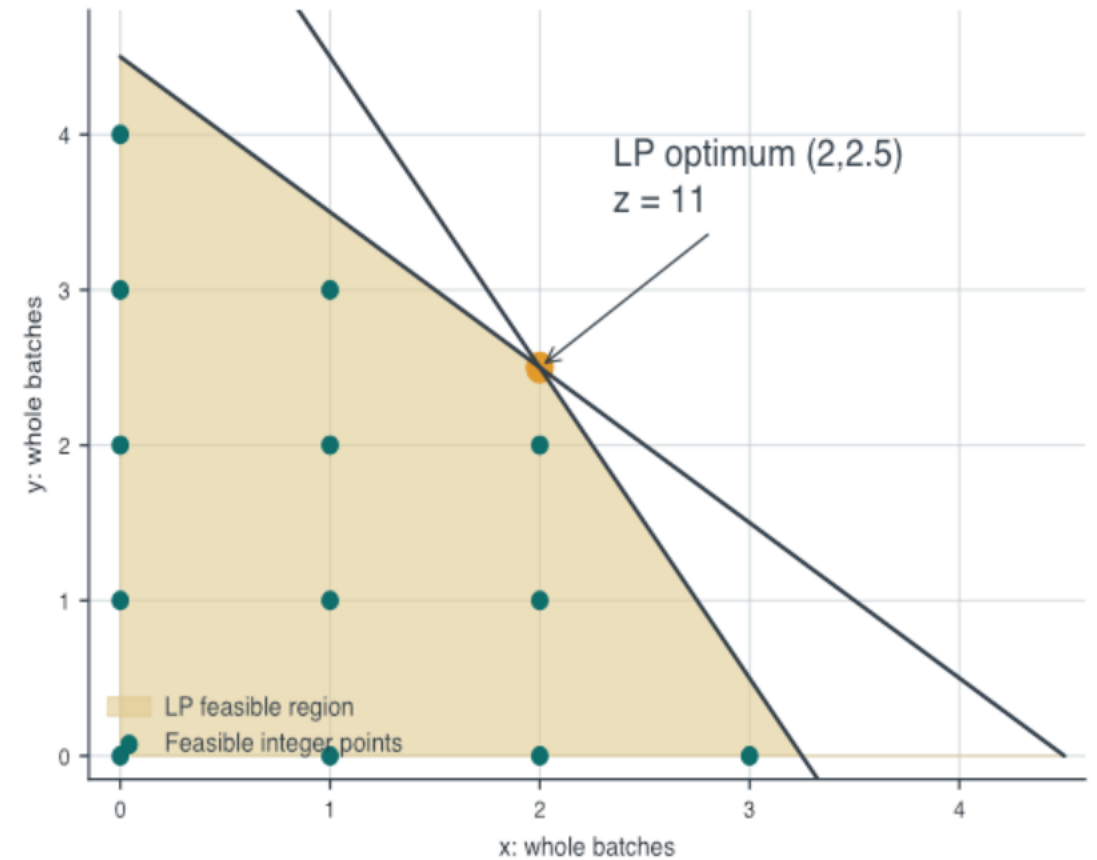


Solve the continuous relaxation

Intersection: $x + y = 4.5$ and $2x + y = 6.5$.

Subtract: $x = 2$; then $y = 2.5$.

Add constraints: $3x + 2y \leq 11$. This point attains 11.

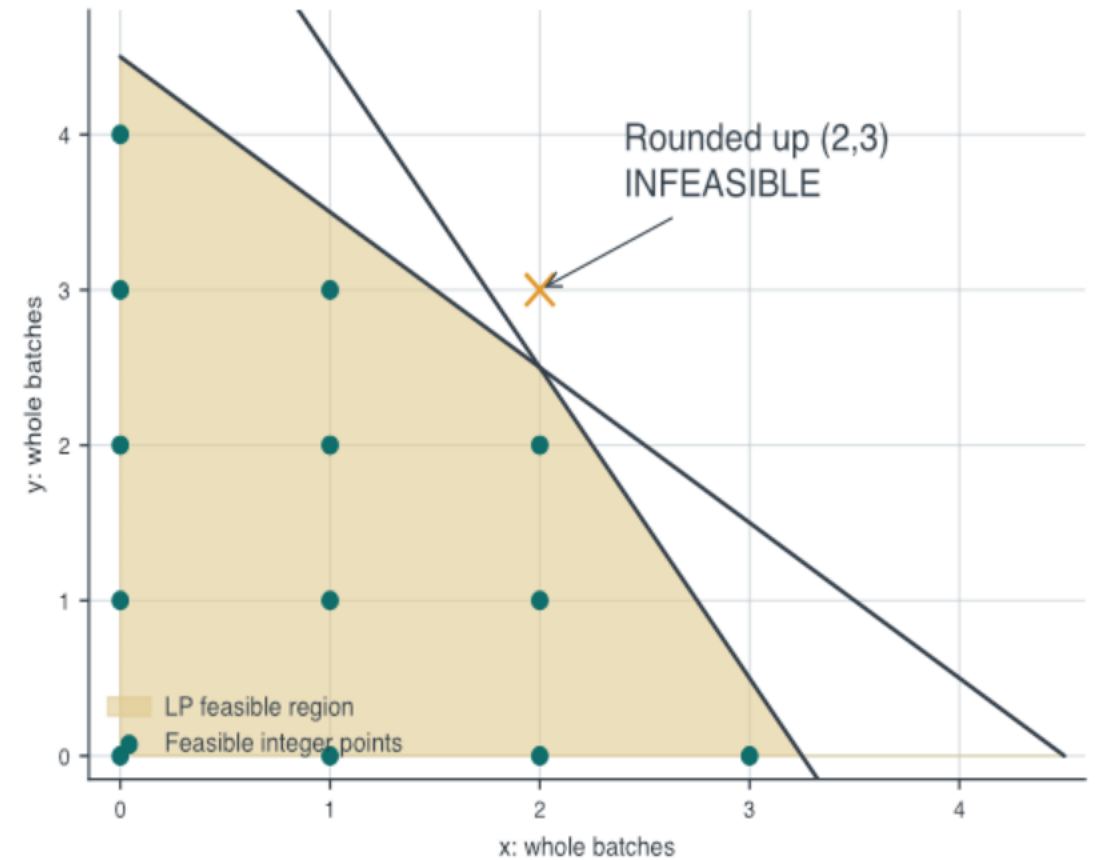


Rounding is not a solution method

Round upward to (2,3): resource use (5,7).

Both limits (4.5,6.5) are violated.

Rounding downward happens to work here; it needs proof.

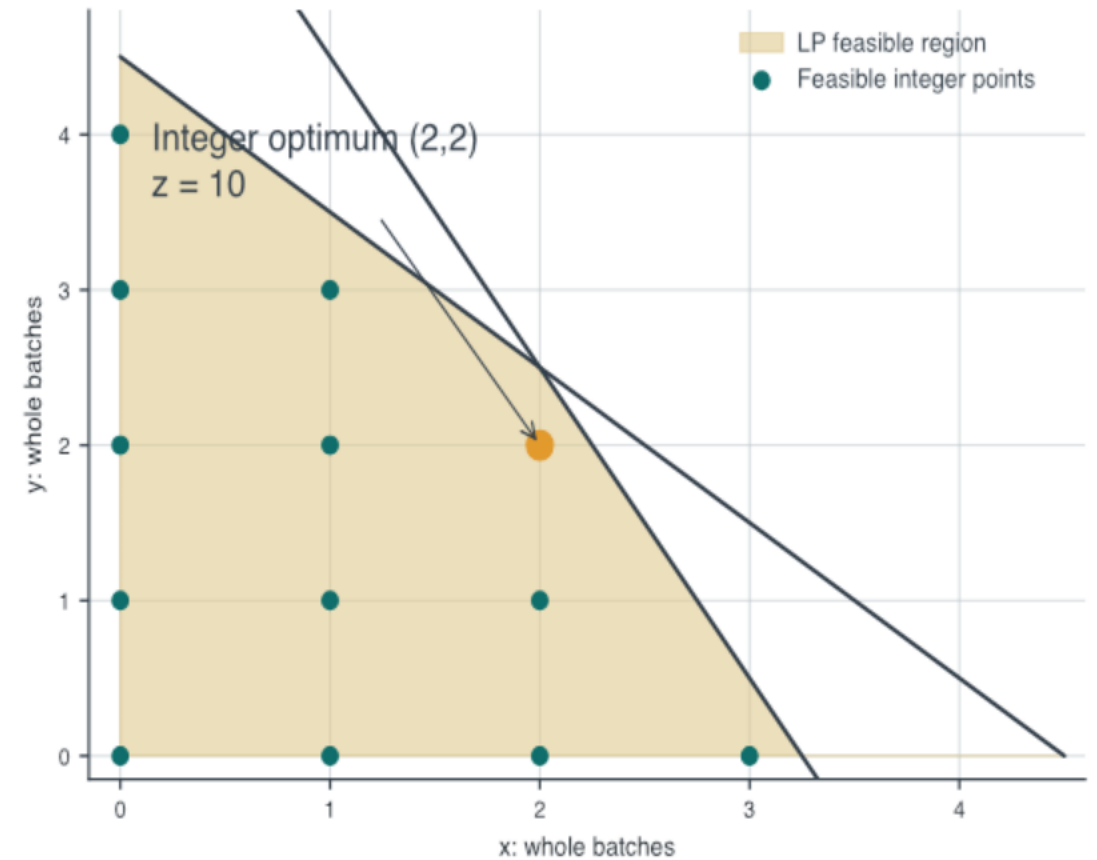


The integer optimum

Enumerate feasible integer points.

Best point: $(x,y) = (2,2)$, value 10.

LP upper bound 11; integer value 10.



Bounds depend on objective direction

Maximization: feasible integer value $\leq$ optimum $\leq$ LP bound.

Minimization: LP bound $\leq$ optimum $\leq$ feasible integer value.

A bound is useful even when its solution is fractional.

Pause: check a proposed batch plan

For $(x,y) = (3,1)$, calculate both resource uses.

Is the plan feasible? What is its objective?

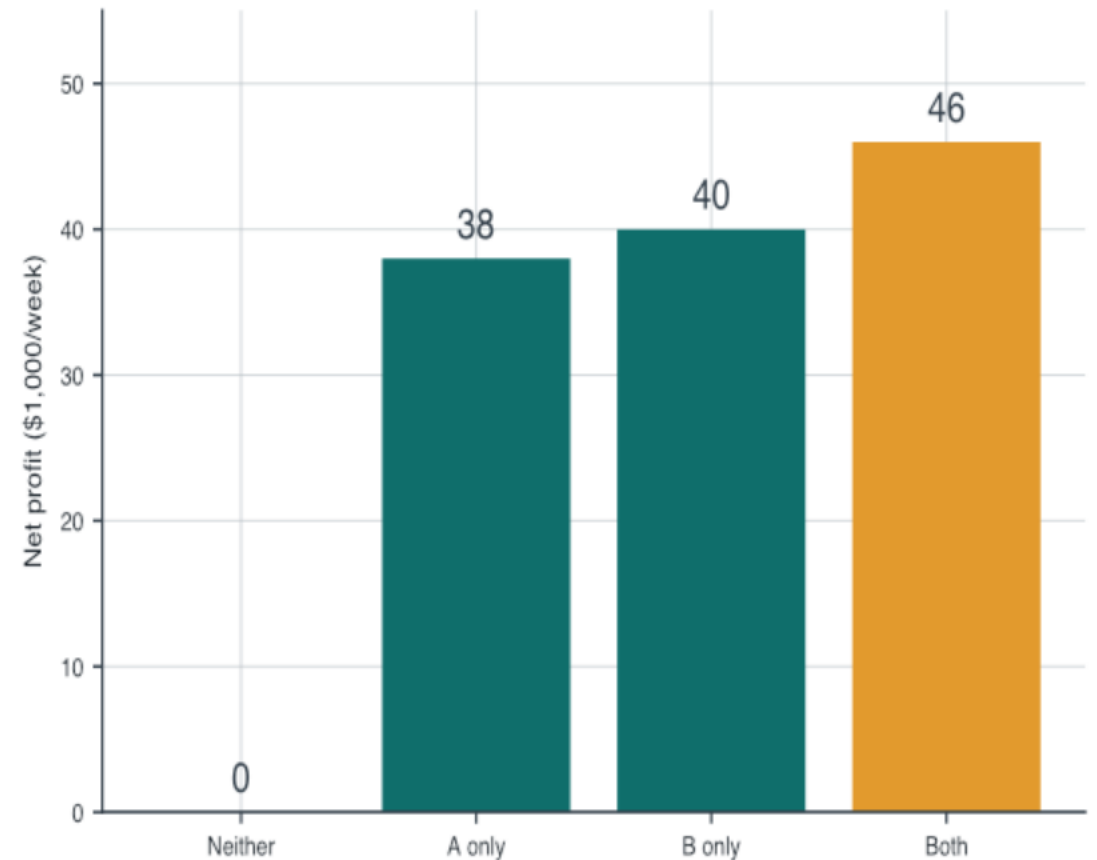
Explain why a high objective is not enough.

A plant with two optional lines

Illustrative weekly data; money in \$1,000/week.

Line A: contribution 8/tonne; opening cost 10.

Line B: contribution 6/tonne; opening cost 8.

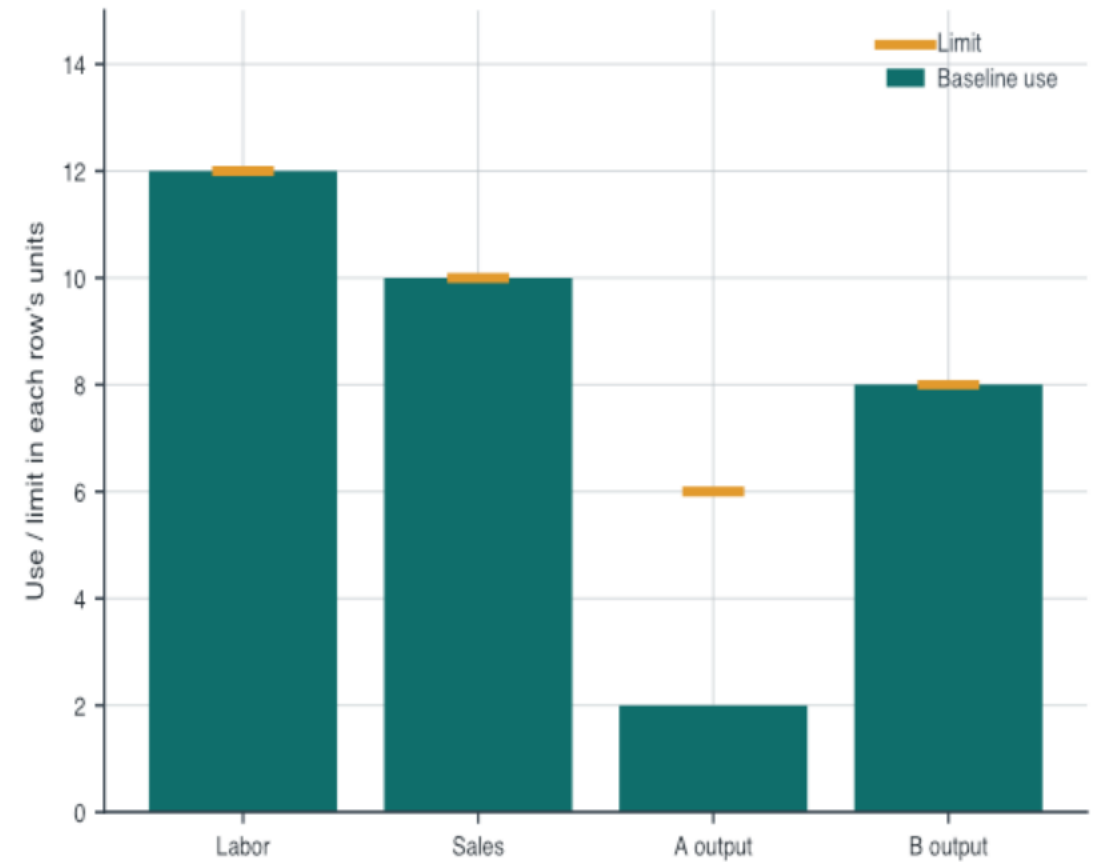


Capacity and resource data

A capacity 6 tonnes/week; B capacity 8.

Labor use: 2 units/tonne for A, 1 for B.

Labor ≤ 12 units/week; total sales ≤ 10 tonnes/week.



Separate production from opening

$a, b \geq 0$: continuous tonnes/week.

$y_A, y_B \in \{0, 1\}$: whether each line is opened.

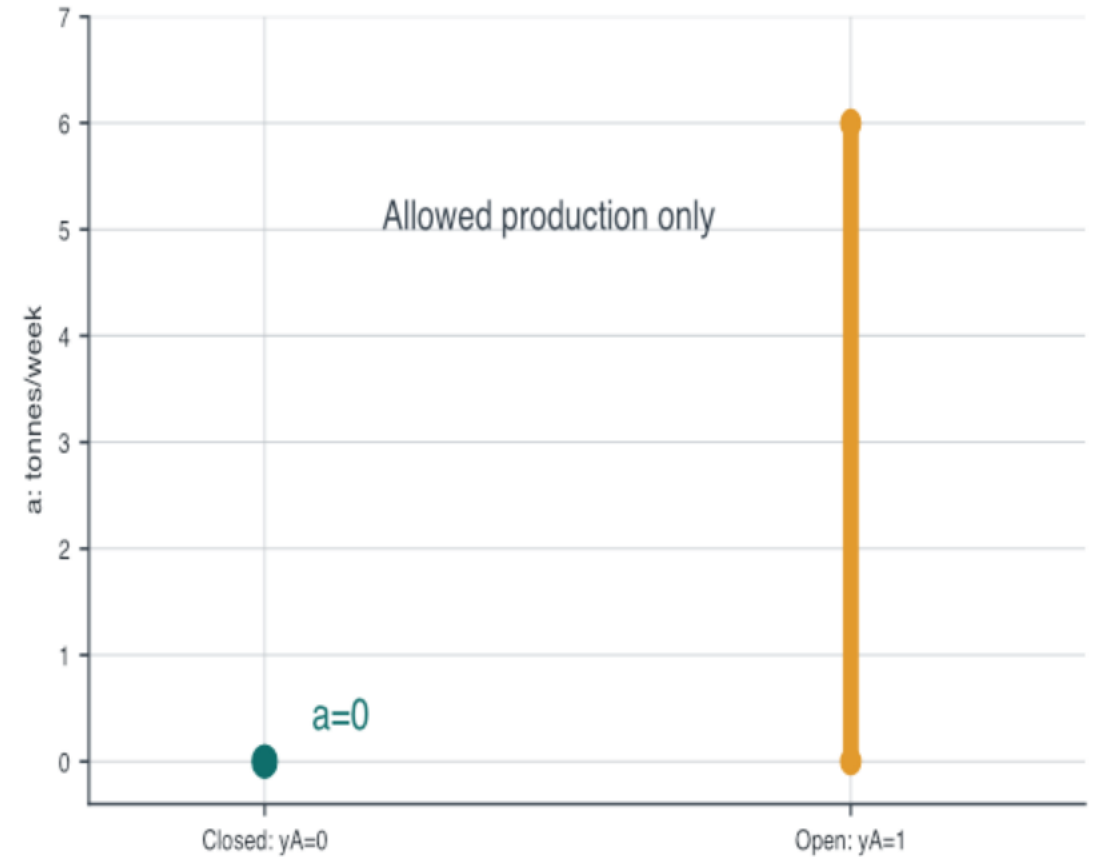
Opening permits production; it does not force full output.

Linking constraint: $a \leq 6y_A$

If $y_A = 0$, then $0 \leq a \leq 0$: $a = 0$.

If $y_A = 1$, then $0 \leq a \leq 6$.

Use $b \leq 8y_B$ for line B.

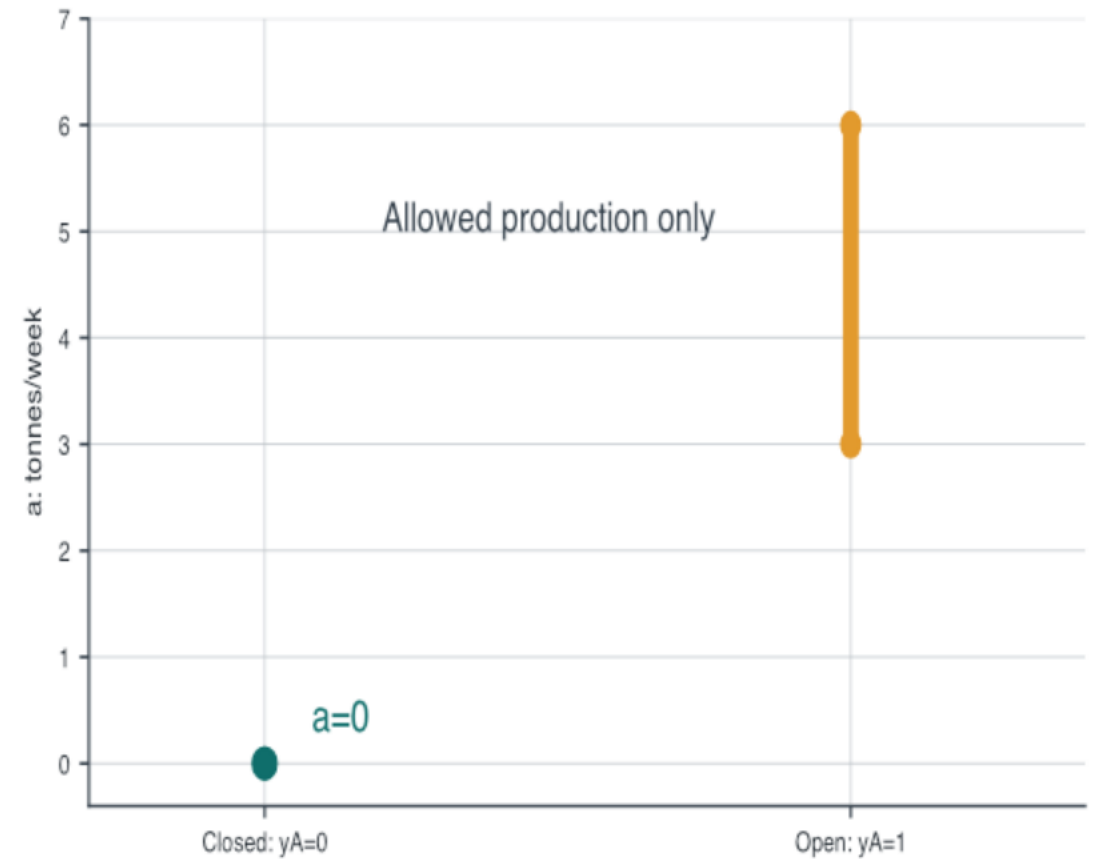


An optional minimum production level

If A opens, require at least 3 tonnes.

Add $a \geq 3y_A$ together with $a \leq 6y_A$.

Closed: $a=0$. Open: $3 \leq a \leq 6$.



Logic with binary variables

At most one line: $y_A + y_B \leq 1$.

Exactly one line: $y_A + y_B = 1$.

A requires B: $y_A \leq y_B$.

The complete baseline MILP

Maximize $8a + 6b - 10y_A - 8y_B$.

$2a + b \leq 12$; $a + b \leq 10$.

$a \leq 6y_A$; $b \leq 8y_B$; $a, b \geq 0$; y_A, y_B binary.

$$\max 8a + 6b - 10y_A - 8y_B$$

$$2a + b \leq 12, \quad a + b \leq 10$$

$$a \leq 6y_A, \quad b \leq 8y_B$$

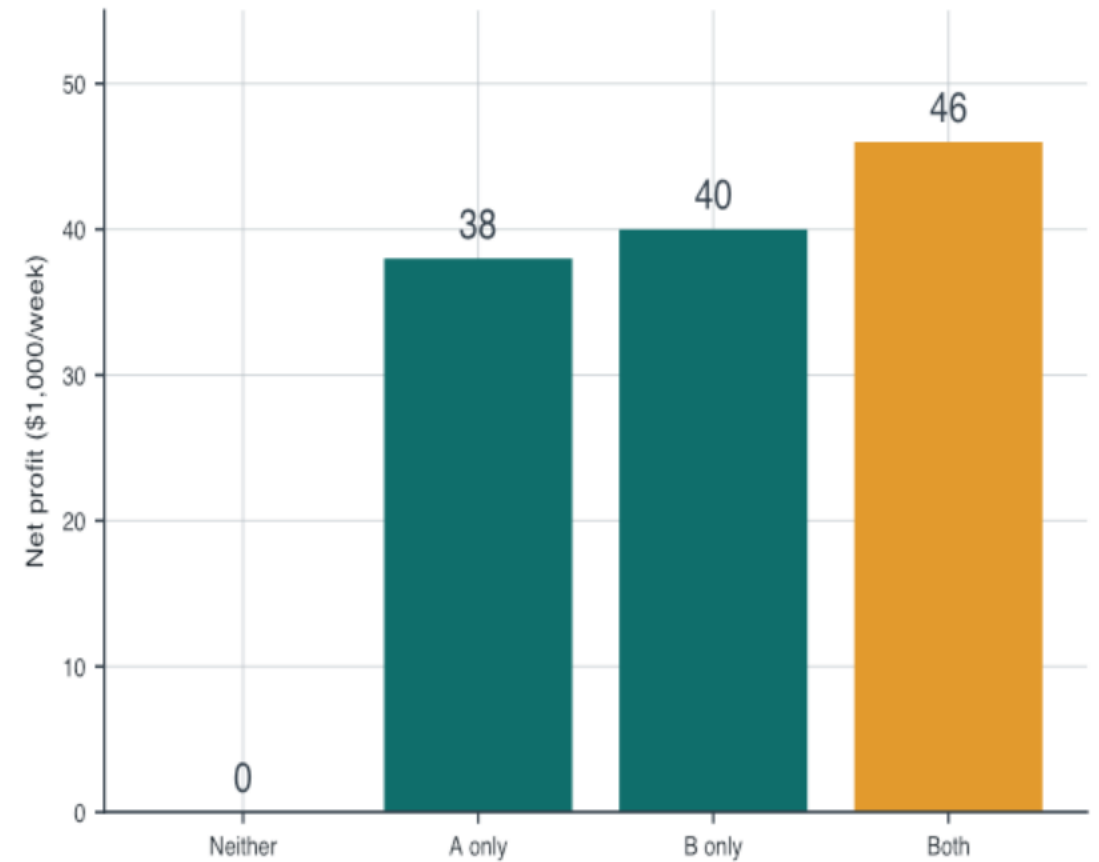
$$a, b \geq 0, \quad y_A, y_B \in \{0, 1\}$$

Verify by enumerating opening choices

Neither: profit 0. A only: $a=6$, profit 38.

B only: $b=8$, profit 40.

Both: $a=2, b=8$, profit 46.

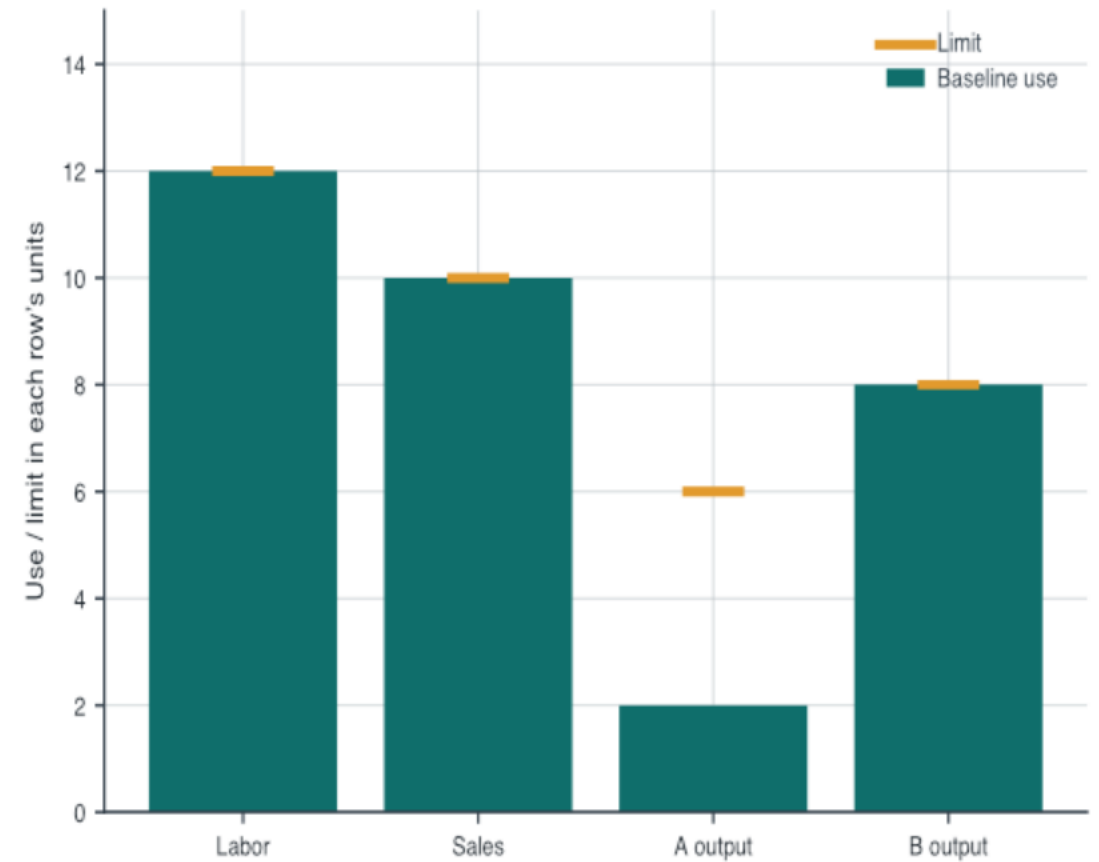


Audit the chosen production plan

$a=2, b=8, y_A=y_B=1$.

Labor: $2(2)+8=12$. Sales: $2+8=10$.

Profit: $16+48-10-8=46$.

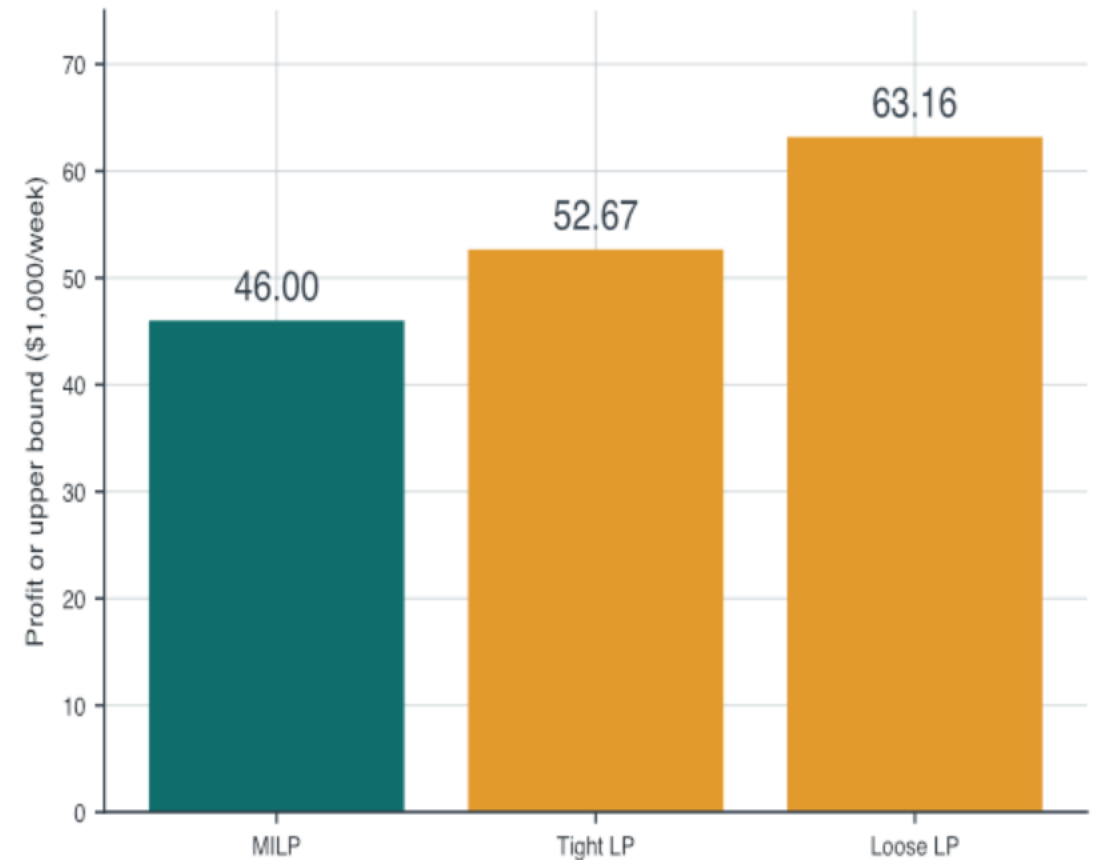


Choose a meaningful upper bound

Use the physical bound $U=6$ for $a \leq U y_A$.

An unnecessarily large U weakens the LP relaxation.

Never choose U so small that valid plans are excluded.

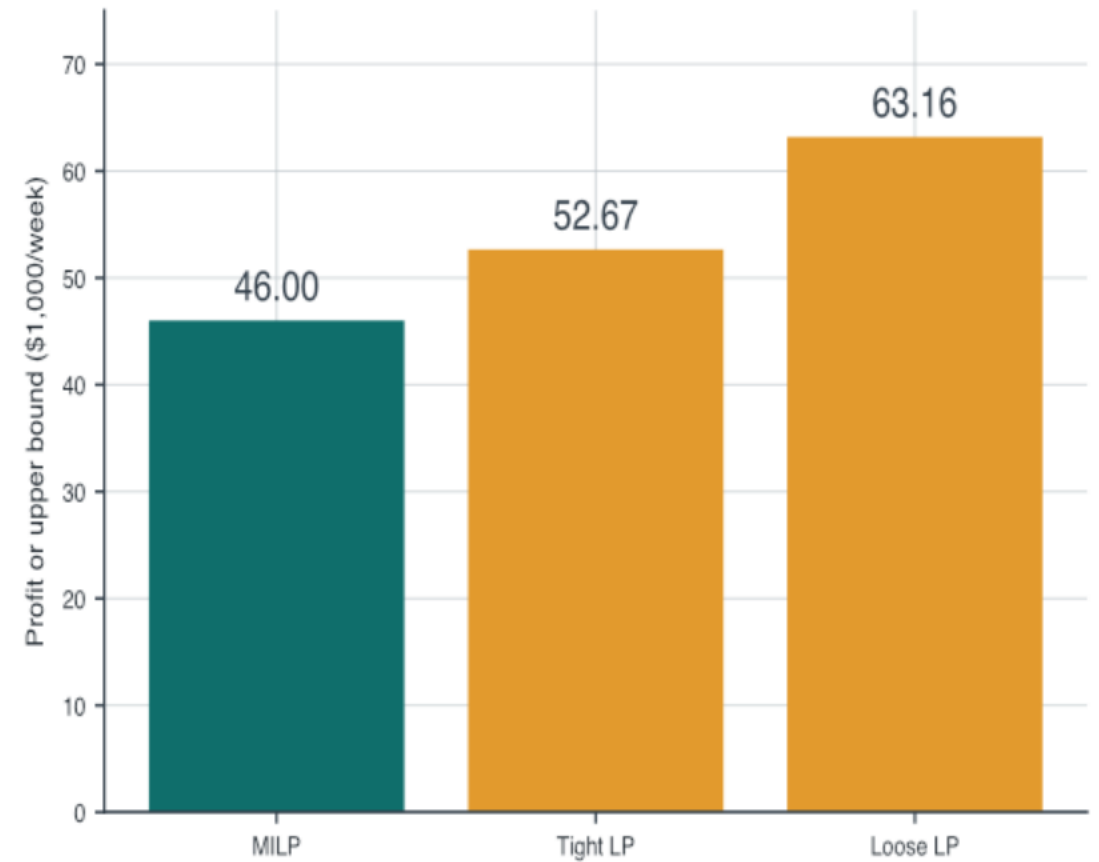


Relax the opening decisions

Allow $0 \leq y_A, y_B \leq 1$ temporarily.

Relaxed result: $a=2, b=8, y_A=1/3, y_B=1$.

Bound 52.667 versus integer profit 46.



A loose bound makes the relaxation weaker

Keep $a \leq 6$ and $b \leq 8$ as physical limits.

Use $a \leq 100y_A$ and $b \leq 100y_B$ only for linking.

Relaxed profit becomes 63.16; the MILP stays at 46.



Branch-and-bound: maintain two values

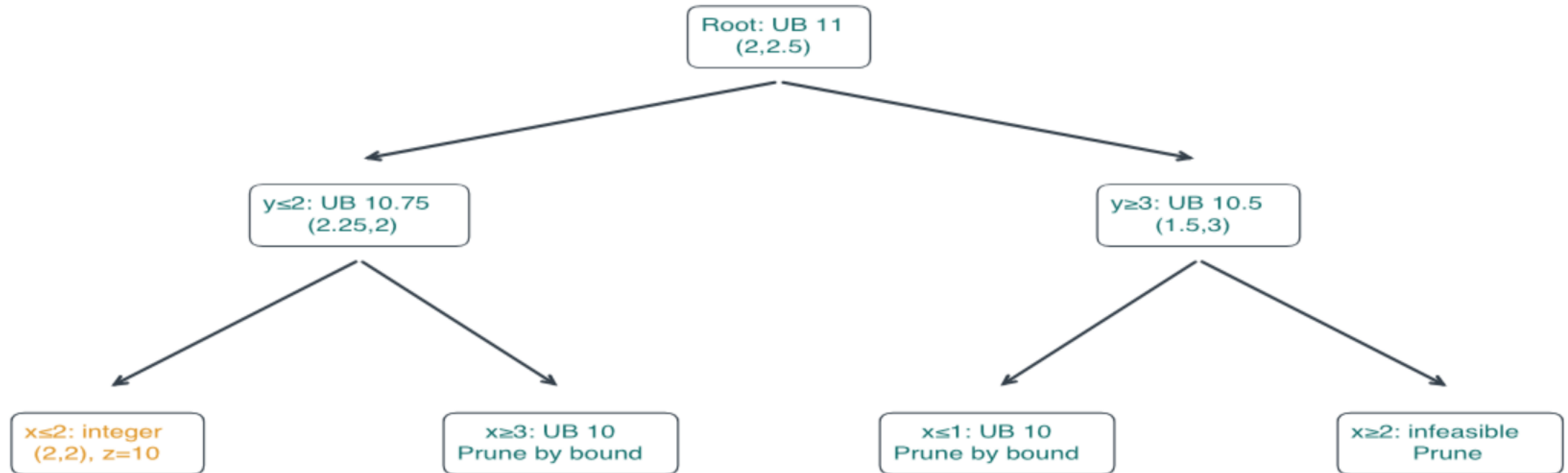
Incumbent: best feasible integer plan found so far.

Node bound: best value of a restricted LP relaxation.

For max: prune if $\text{bound} \leq \text{incumbent}$, when seeking one optimum.

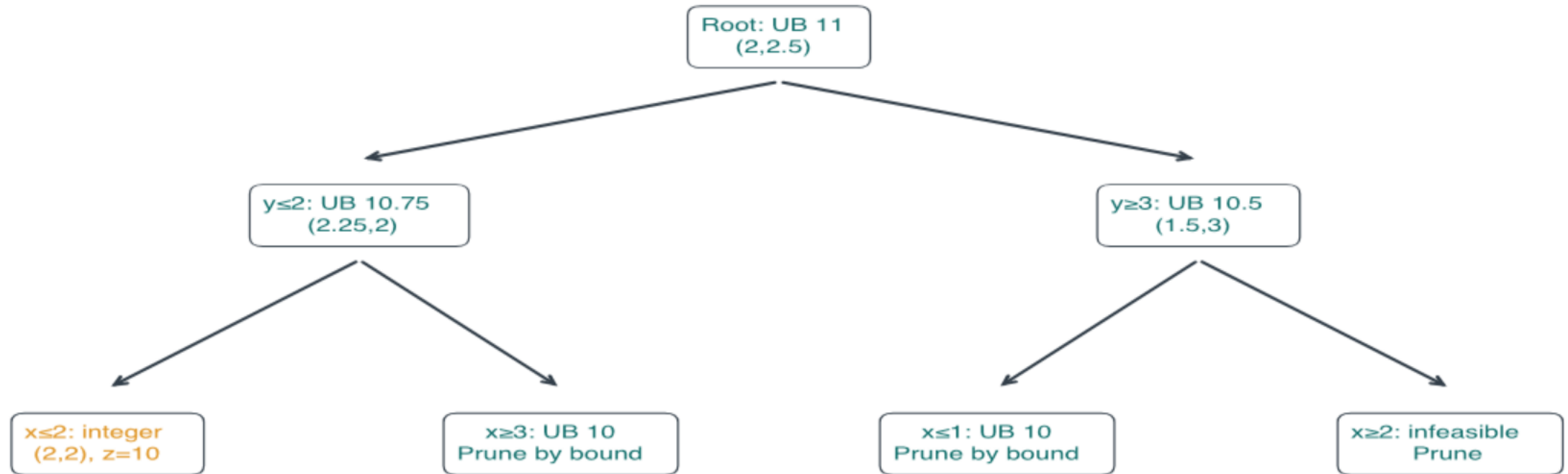
Root node: split a fractional variable

Batch LP: $(x,y)=(2,2.5)$, upper bound 11. • Every integer y satisfies $y \leq 2$ or $y \geq 3$. • These two branches cover every integer possibility.



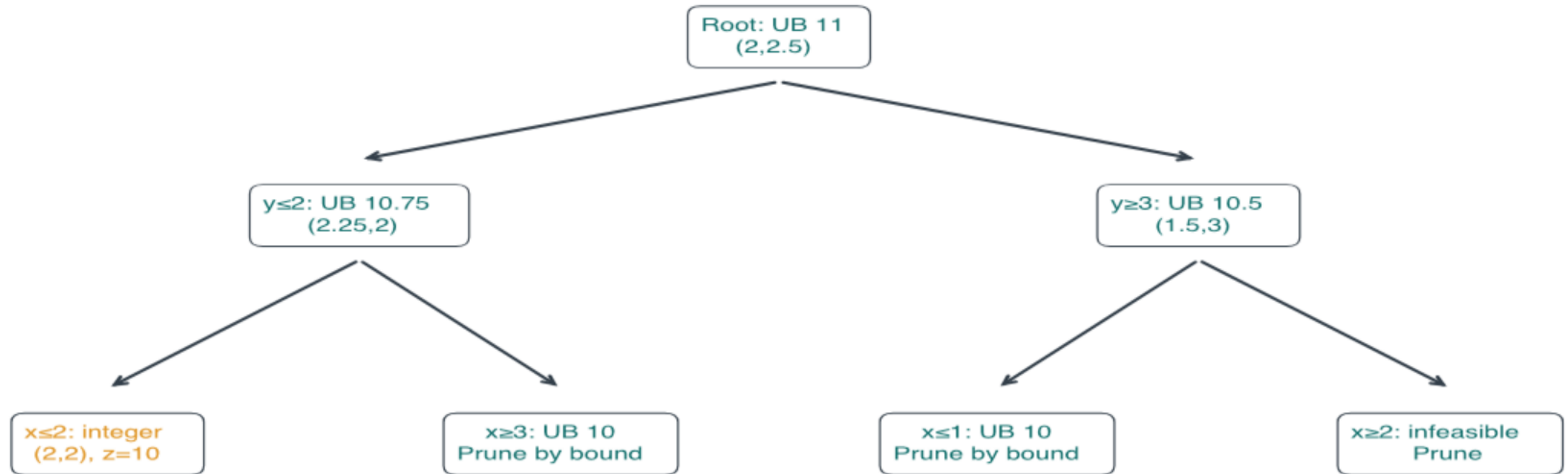
Left branch: $y \leq 2$

LP result: (2.25,2), upper bound 10.75. • x is fractional, so split $x \leq 2$ or $x \geq 3$. • In the $x \leq 2$ child: (2,2), value 10, is integer.



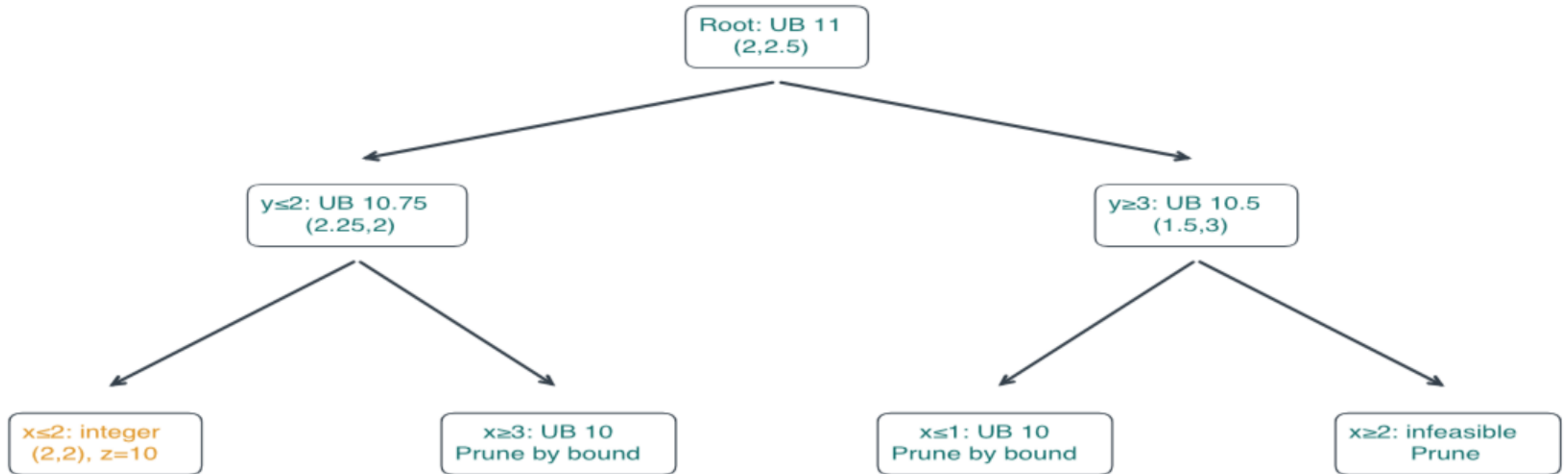
Prune the other left child

Constraints include $y \leq 2$ and $x \geq 3$. • Its LP optimum is $(3, 0.5)$, bound 10. • Bound equals incumbent 10: no better value is possible.



Right branch: $y \geq 3$

LP result: (1.5,3), upper bound 10.5. • Split $x \leq 1$ or $x \geq 2$. • $x \leq 1$ gives bound 10; $x \geq 2$ is infeasible.



What certifies completion?

Feasible incumbent value: 10.

Every remaining branch is closed or bounded by 10.

Therefore the global integer optimum is 10.

Stopping early and reading the gap

A feasible plan may exist before optimality is proved.

For maximization, lower bound = incumbent value.

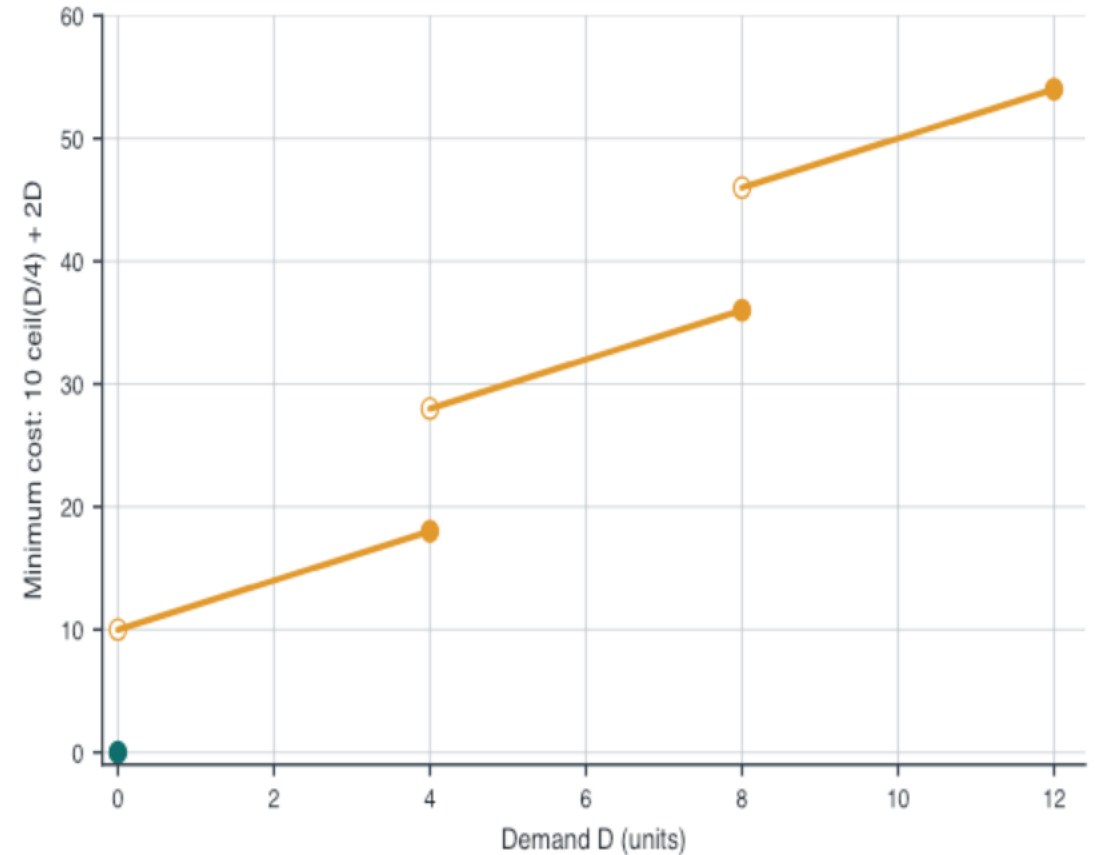
Negate a minimization dual bound to get a max upper bound.

Why LP shadow prices need care

Separate example: each machine supplies 4 units and costs 10.

Variable cost = 2 per unit: $C(D) = 10 \text{ ceil}(D/4) + 2D$.

For $D \geq 0$, machine counts jump. Re-solve MILP after changes.



Python lab: predict before running

Download MILP-guided-lab.ipynb.

Use Python with NumPy and SciPy (version 1.9 or later).

Run cells in order; use the separate exercise sheet for practice.

Step 1: choose a fixed variable order

$v = (a, b, yA, yB).$

Minimization coefficients: $c = (-8, -6, 10, 8).$

Return to profit with $\text{profit} = -\text{result.fun}.$

$$A = \begin{array}{c} \begin{array}{cccc} a & b & yA & yB \end{array} \\ \left[\begin{array}{cccc} 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & -6 & 0 \\ 0 & 1 & 0 & -8 \end{array} \right] \end{array}$$

$$A v \leq (12, 10, 0, 0)^T$$

Step 2: encode variable domains

integrality = [0,0,1,1].

Lower bounds = [0,0,0,0].

Upper bounds = [6,8,1,1].

Step 3: build the constraint matrix

Rows: labor, sales, link A, link B.

$$A v \leq [12, 10, 0, 0].$$

LinearConstraint(A, $-\infty$, upper) encodes upper bounds.

$$A = \begin{array}{c} \begin{array}{cccc} a & b & yA & yB \end{array} \\ \left[\begin{array}{cccc} 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & -6 & 0 \\ 0 & 1 & 0 & -8 \end{array} \right] \end{array}$$

$$A v \leq (12, 10, 0, 0)^T$$

Step 4: solve and check status

```
result = milp(c, integrality=..., bounds=..., constraints=...).
```

Check success, status, and message.

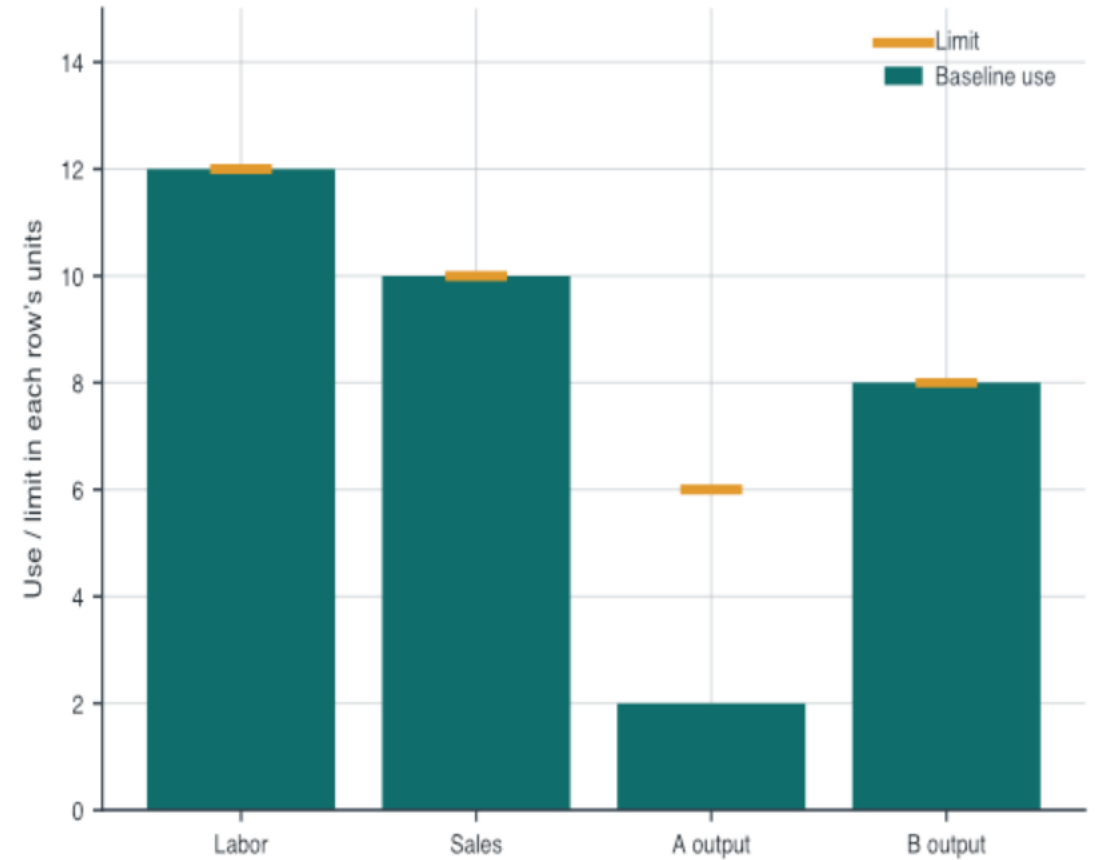
Baseline: $v=(2,8,1,1)$, profit 46.

Step 5: audit every constraint

Compute `resource_use = A @ result.x`.

Check `resource_use ≤ upper` within a small tolerance.

Check binary values and recompute profit independently.



Step 6: compare relaxation and MILP

Relaxed profit: 52.667; MILP profit: 46.

Relaxed $y_A=1/3$ is not an opening decision.

Enumerate four binary patterns to verify the MILP.

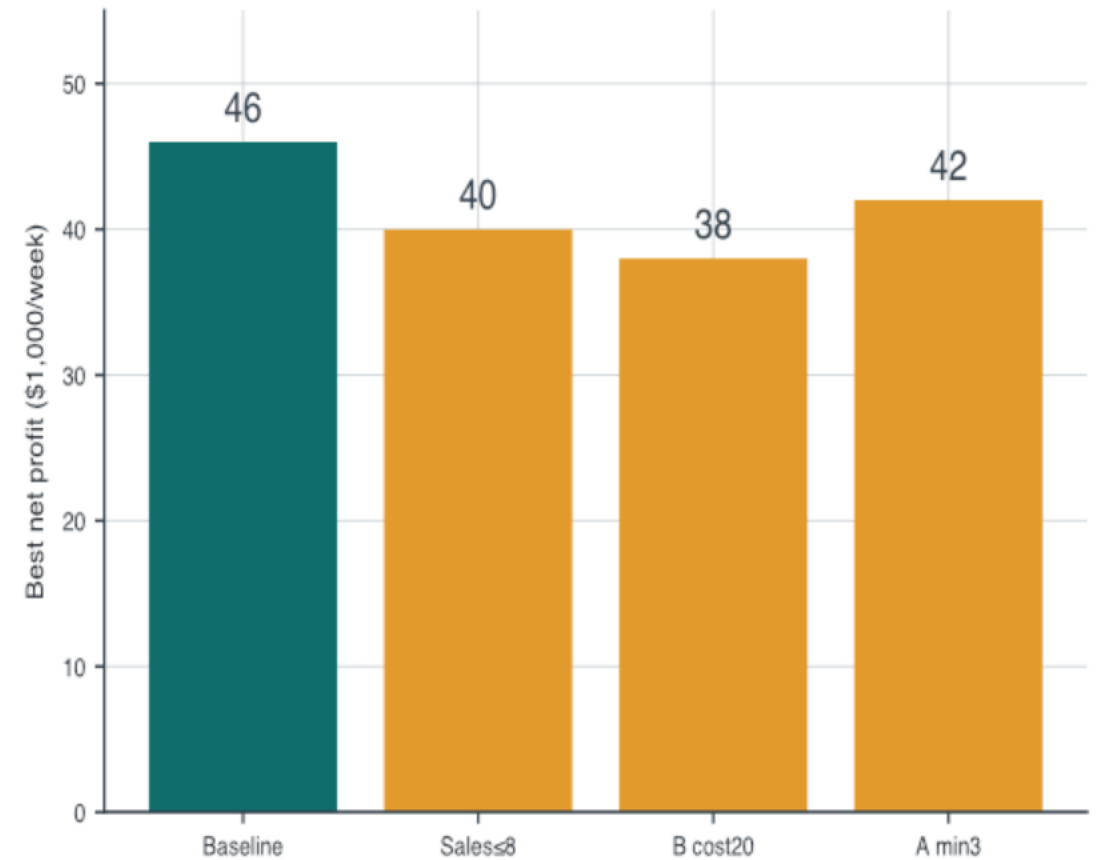


Step 7: change one assumption

Sales cap 8: open B only, profit 40.

B opening cost 20: open A only, profit 38.

A minimum batch 3: both open, $a=3, b=6$, profit 42.

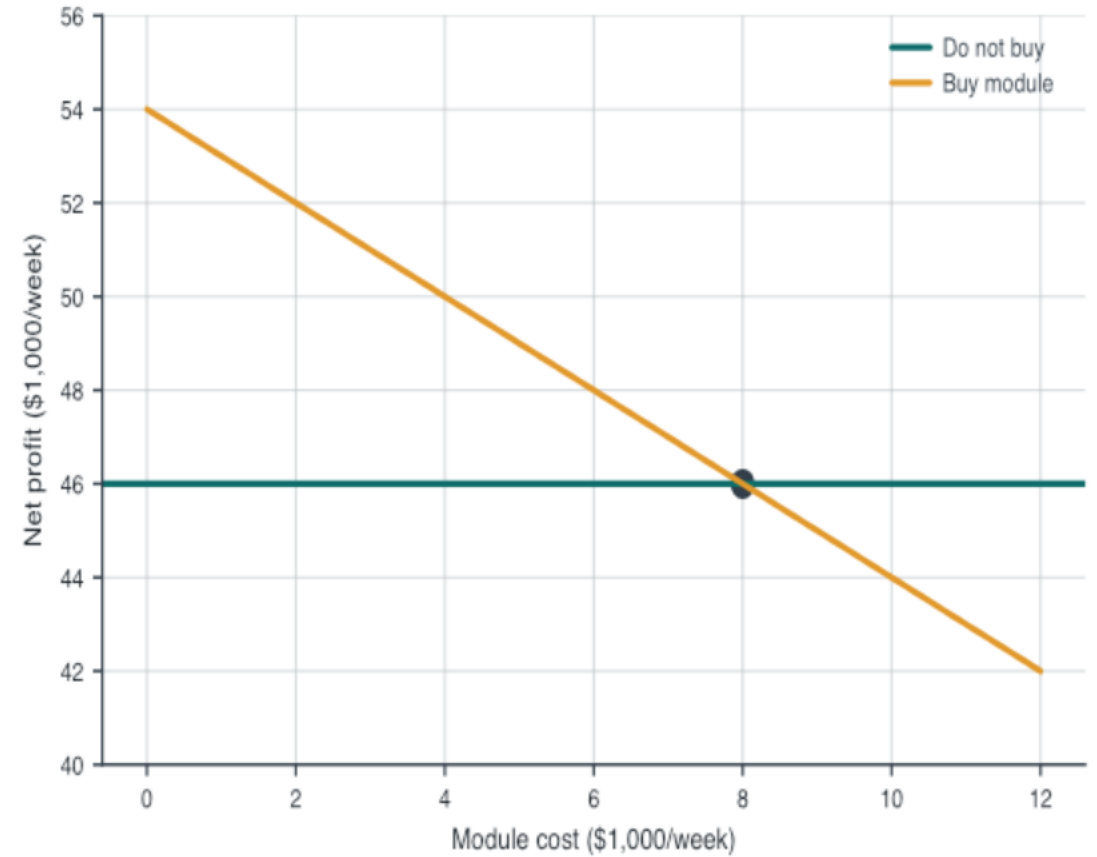


Step 8: consider an extra capacity module

$k \in \{0, 1\}$; labor capacity = $12 + 4k$.

Module cost = 7; subtract $7k$ from profit.

Optimum: $k=1, a=6, b=4$; net profit 47.

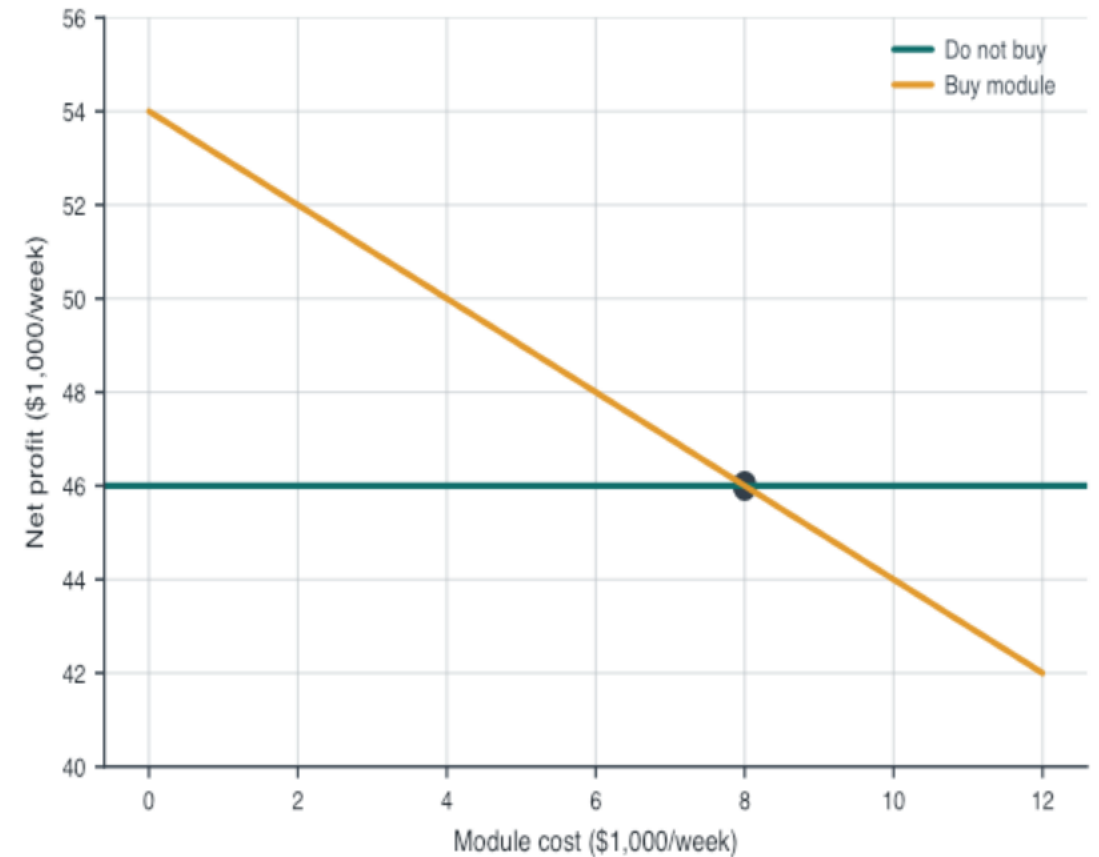


Step 9: compare purchase costs

Without purchase: profit 46.

With module before module cost: profit 54.

Cost 7 $\rightarrow$ buy; cost 10 $\rightarrow$ do not buy.



Independent exercises

1. Batch model: reduce the first resource limit to 4.
2. Plant model: reduce the sales cap to 6.
3. Add an at-most-one-line policy to the baseline.

Independent exercises: decisions and bounds

4. Module cost 8: compare buying and not buying.
5. A minimum batch 4: find the best opening pattern.
6. With incumbent 46 and max bound 48, what is certified?

Exercise solutions: batch and sales cap

1. First resource limit 4: unique best (2,2), value 10.
2. Sales cap 6: A only, $a=6$, profit 38.
3. At most one line: B only, $b=8$, profit 40.

Exercise solutions: module and minimum batch

- 4. Cost 8: buy or not buy both give profit 46.
- 5. A minimum batch 4: choose B only, profit 40.
- 6. $46 \leq \text{optimum} \leq 48$; possible improvement ≤ 2 .

Exit Ticket: explain, then submit

Complete four questions on the online form.

Use your own calculations and reasoning.

Submit before opening the following solution slides.



Exit Ticket questions

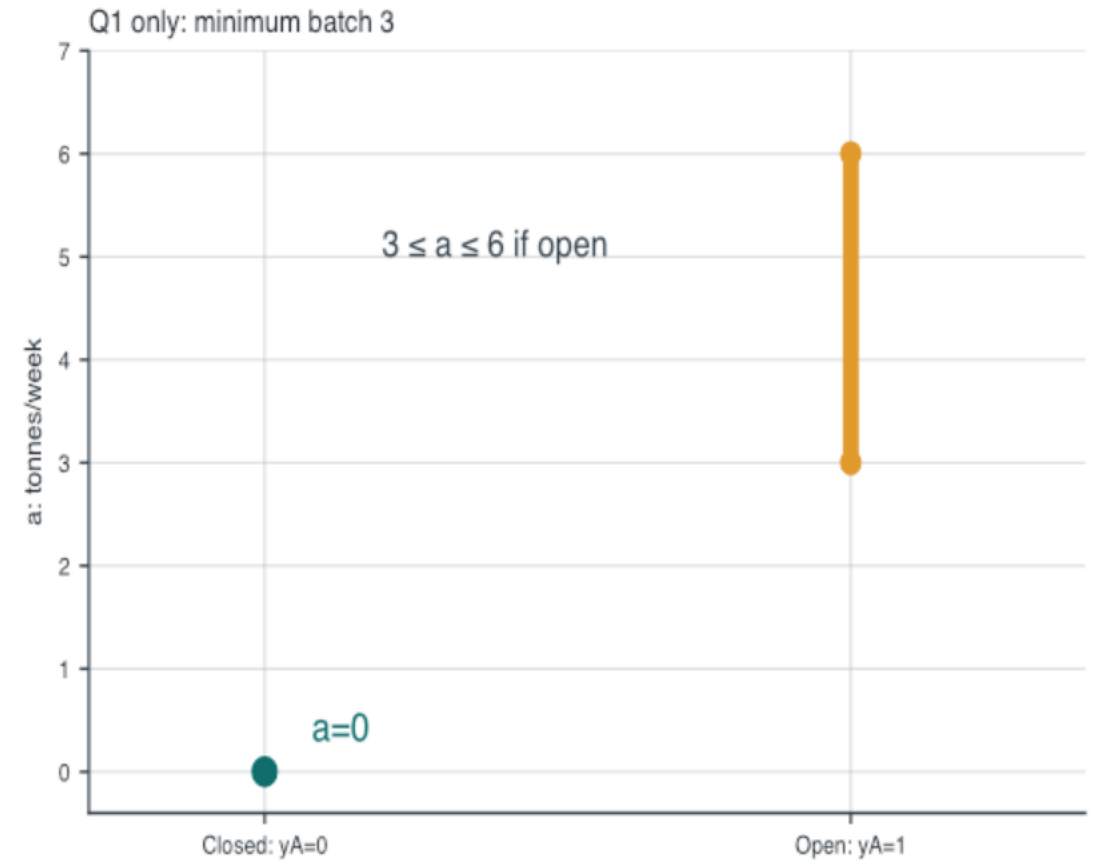
1. Interpret $a \leq 6y_A$ and $a \geq 3y_A$ for binary y_A .
2. Why does relaxed profit 52.667 not beat MILP profit 46?
3. Module cost 9: buy or not? Show net benefit.
4. Incumbent 46: prune nodes with bounds 45,46,48?

Exit Ticket solutions: linking and relaxation

Q1: $yA=0 \Rightarrow a=0$; $yA=1 \Rightarrow 3 \leq a \leq 6$.

Q2 resets to baseline: relaxed $yA=1/3$ pays partial fixed cost.

52.667 is an upper bound, not an attainable MILP profit.



Exit Ticket solutions: purchase and pruning

3. Extra operating benefit $54 - 46 = 8$; cost 9: net -1 .

Do not buy; best total profit remains 46.

4. Prune bounds 45 and 46; keep bound 48 for further work.

A reliable MILP workflow

Define decisions, units, domains, and valid bounds.

Solve; audit feasibility, costs, status, and gap.

Change assumptions one at a time and re-optimize.

Resources and next steps

SciPy milp: coefficients, integrality, bounds, status and gap.

Course page: videos, lab, practice, solutions and Exit Ticket.

Keep your calculations and questions for the next meeting.