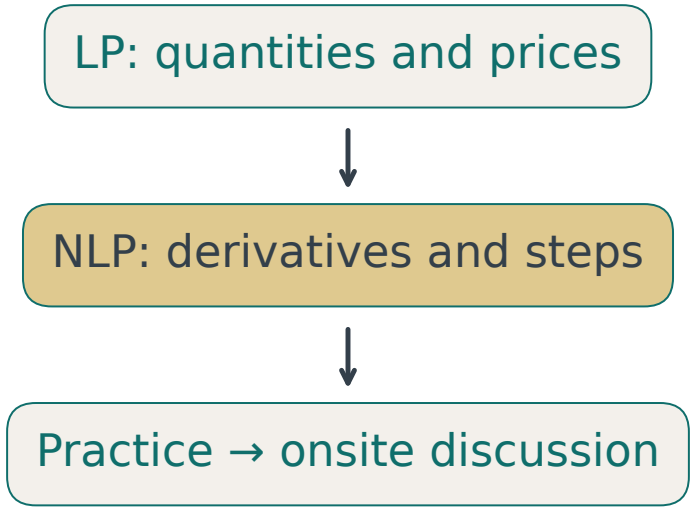
LP and NLP: guided review

Prepare for onsite class

Monday, November 9, 2026

Pause, calculate, then compare your reasoning.

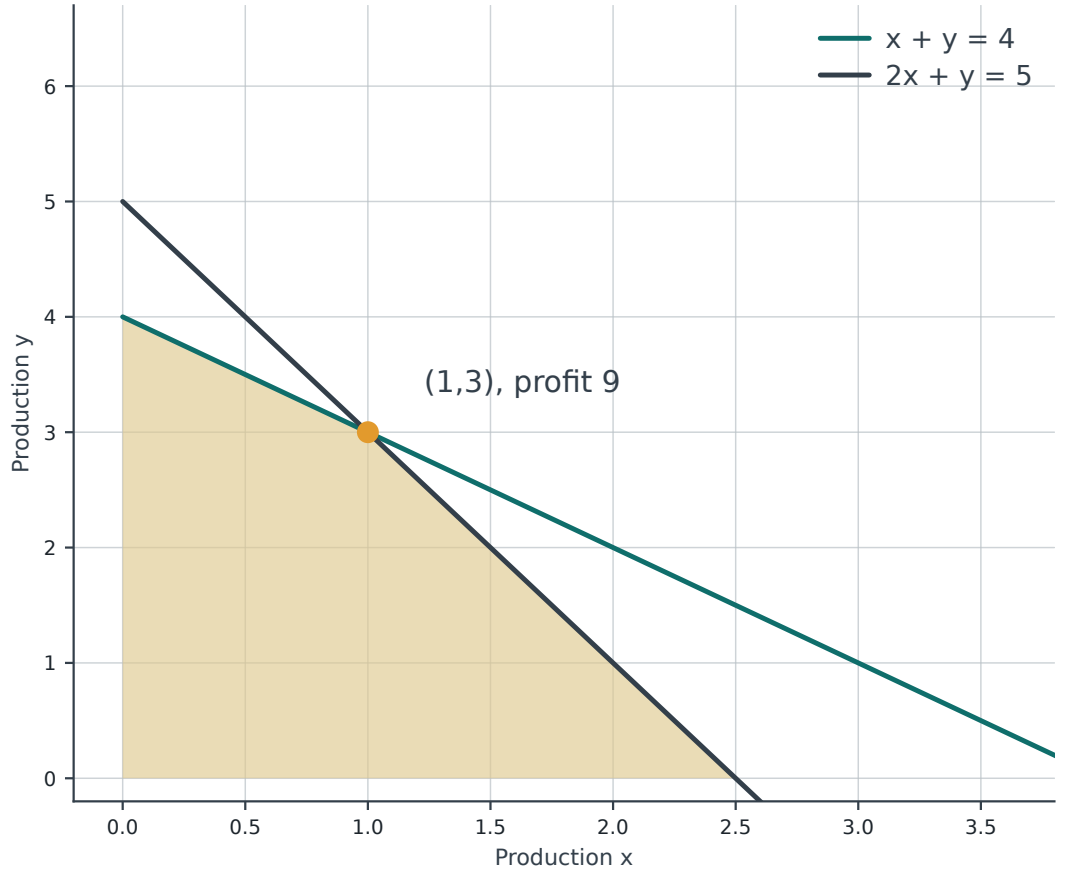


What the primal variables mean

$\max z = 3x + 2y;\ x, y \geq 0$

Resource 1: $x + y \leq 4$

Resource 2: $2x + y \leq 5$

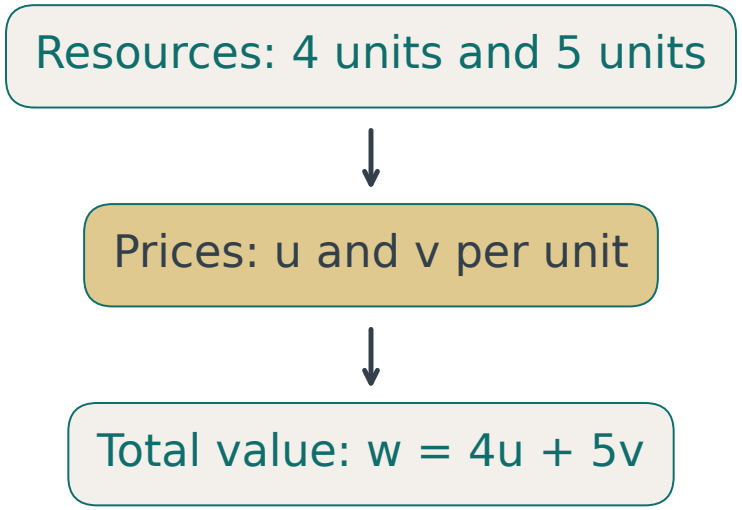


What the dual variables mean

u = price per unit of resource 1

v = price per unit of resource 2

w = 4u + 5v: value of the available resources

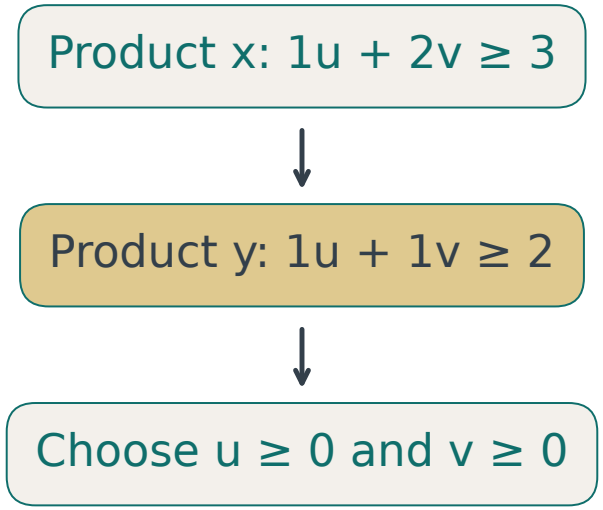


Derive one dual constraint per product

Product x uses (1, 2): cost = u + 2v

Cover its profit: u + 2v ≥ 3

Product y uses (1, 1): u + v ≥ 2

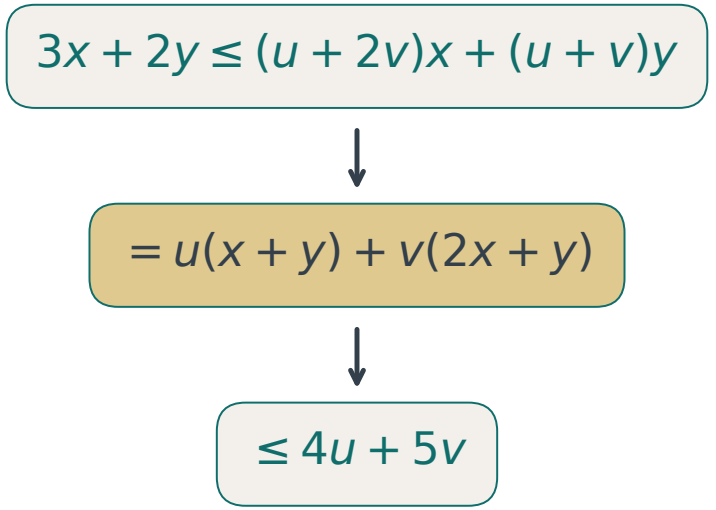


Why the dual gives an upper bound

min w = 4u + 5v; u, v ≥ 0

u + 2v ≥ 3; u + v ≥ 2

For any feasible pair: z ≤ w

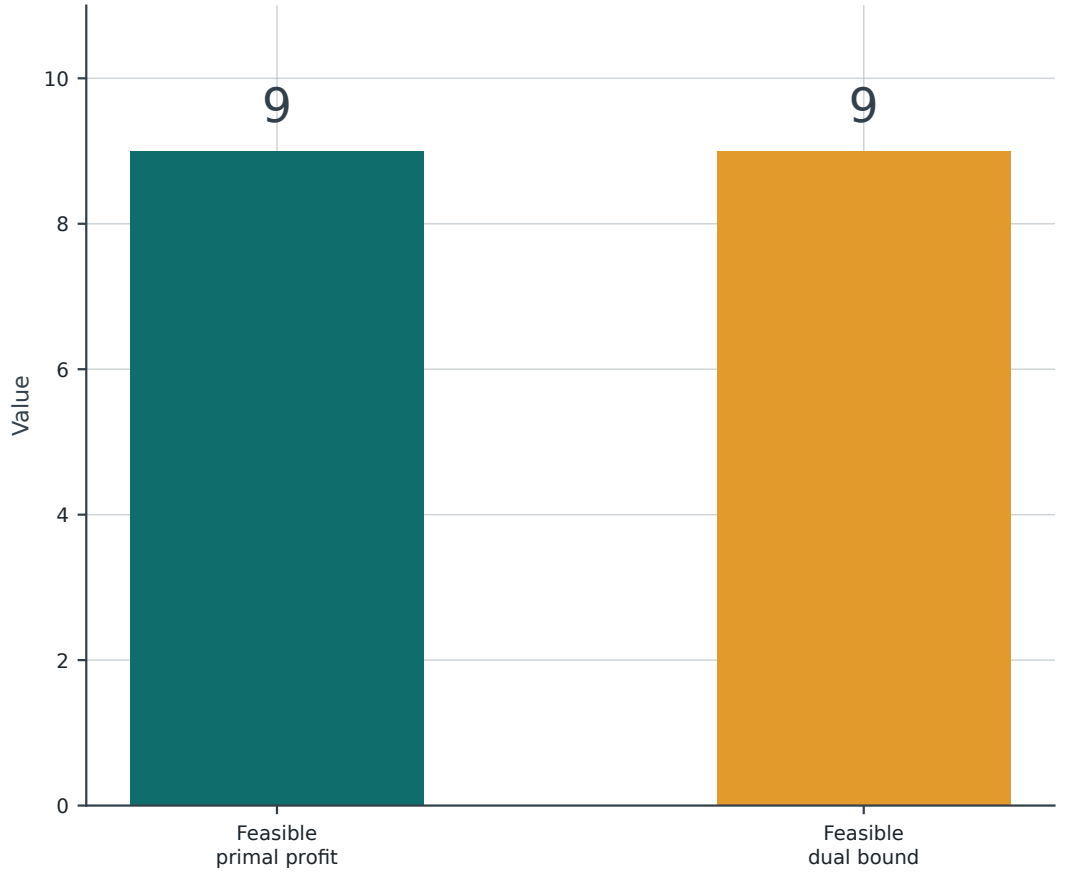


An optimality certificate

Primal (x, y) = (1, 3): z = 9

Dual (u, v) = (1, 1): w = 9

Both are feasible; equal values prove optimality.

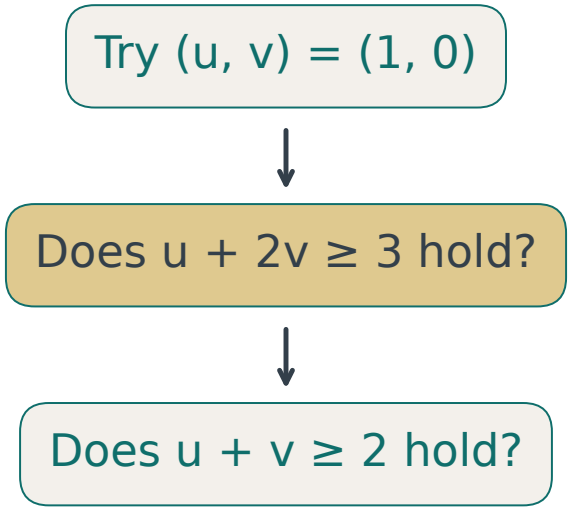


Practice: test a proposed price

Proposed prices: (u, v) = (1, 0)

Check both dual constraints and compute w.

Can w bound every feasible primal profit?

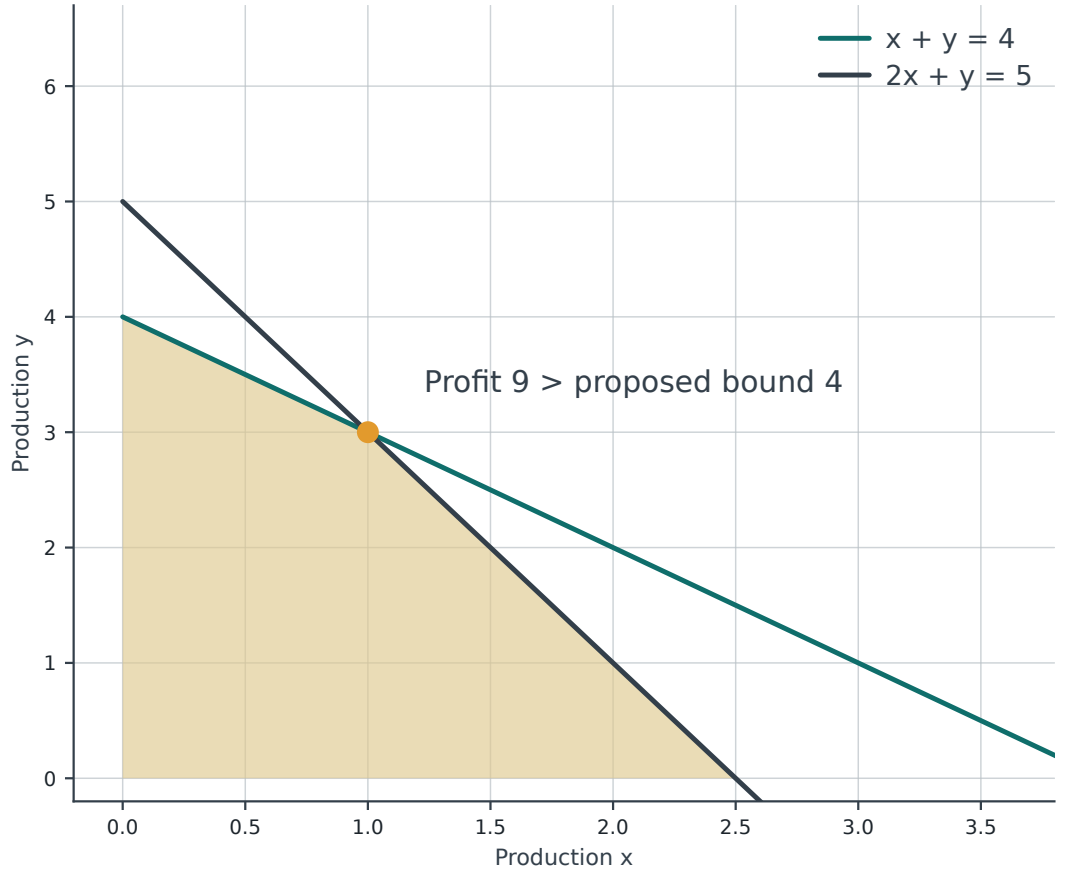


Solution: feasibility comes first

$1 + 2(0) = 1 < 3;\ 1 + 0 = 1 < 2$

$w = 4$, but feasible $(1, 3)$ earns $z = 9$.

The proposed prices are not dual feasible.

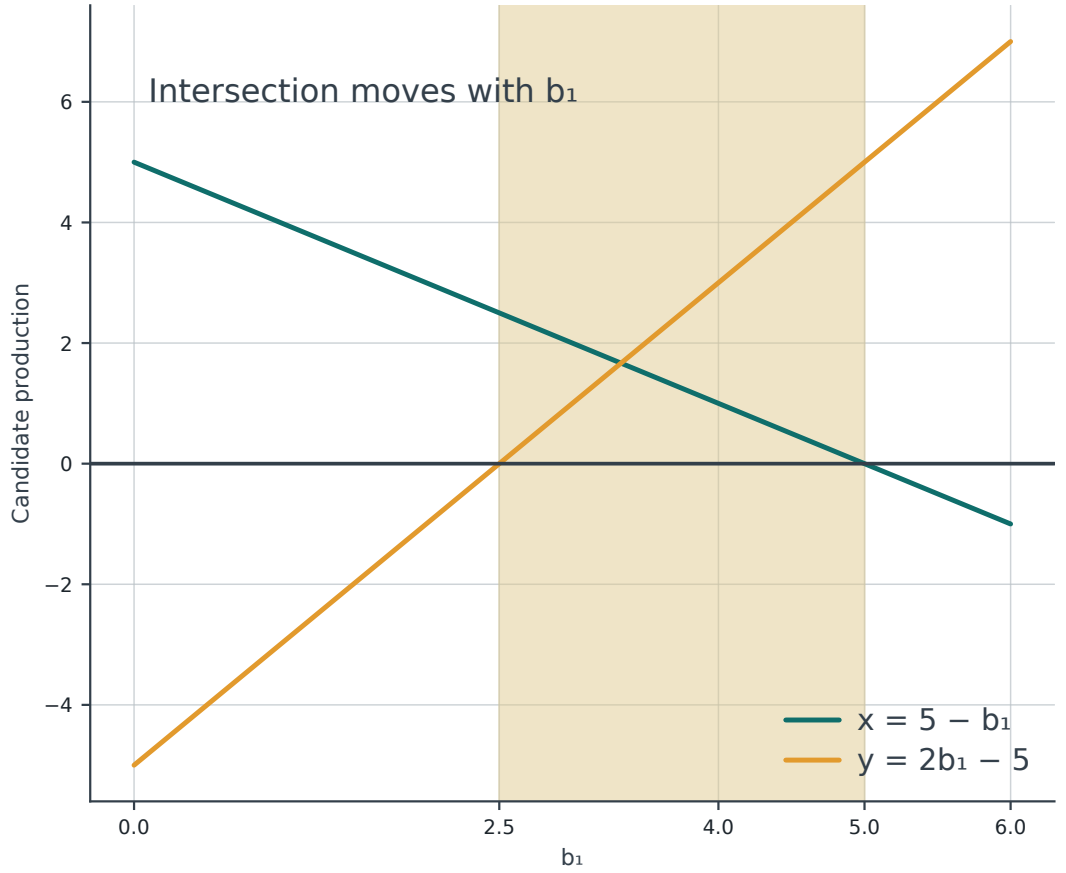


Derive the shadow-price interval

Keep $b_2 = 5$ and change b_1 only.

Solve $x + y = b_1$ and $2x + y = 5$.

$x = 5 - b_1$; $y = 2b_1 - 5$

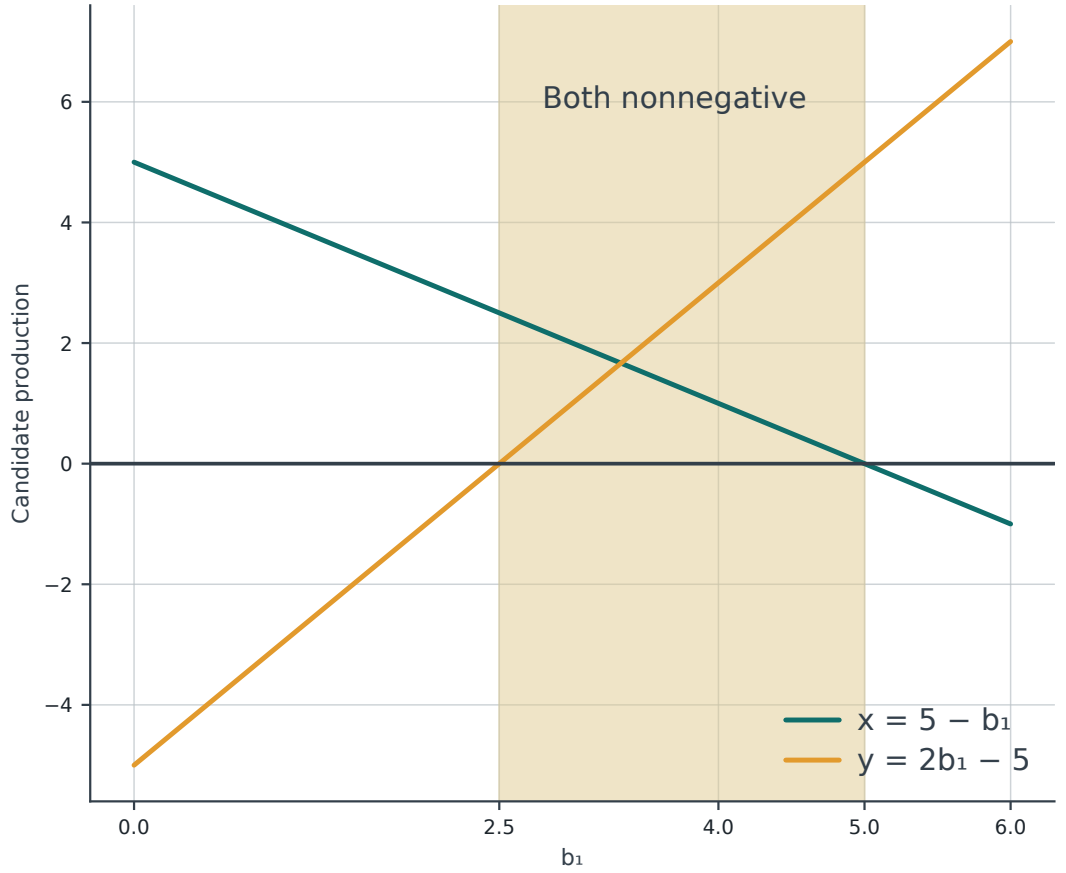


Feasibility determines the interval

$x \geq 0$ requires $b_1 \leq 5$; $y \geq 0$ requires $b_1 \geq 2.5$.

On $2.5 \leq b_1 \leq 5$: primal $z = b_1 + 5$ = dual w.

At the endpoints, use one-sided slopes.

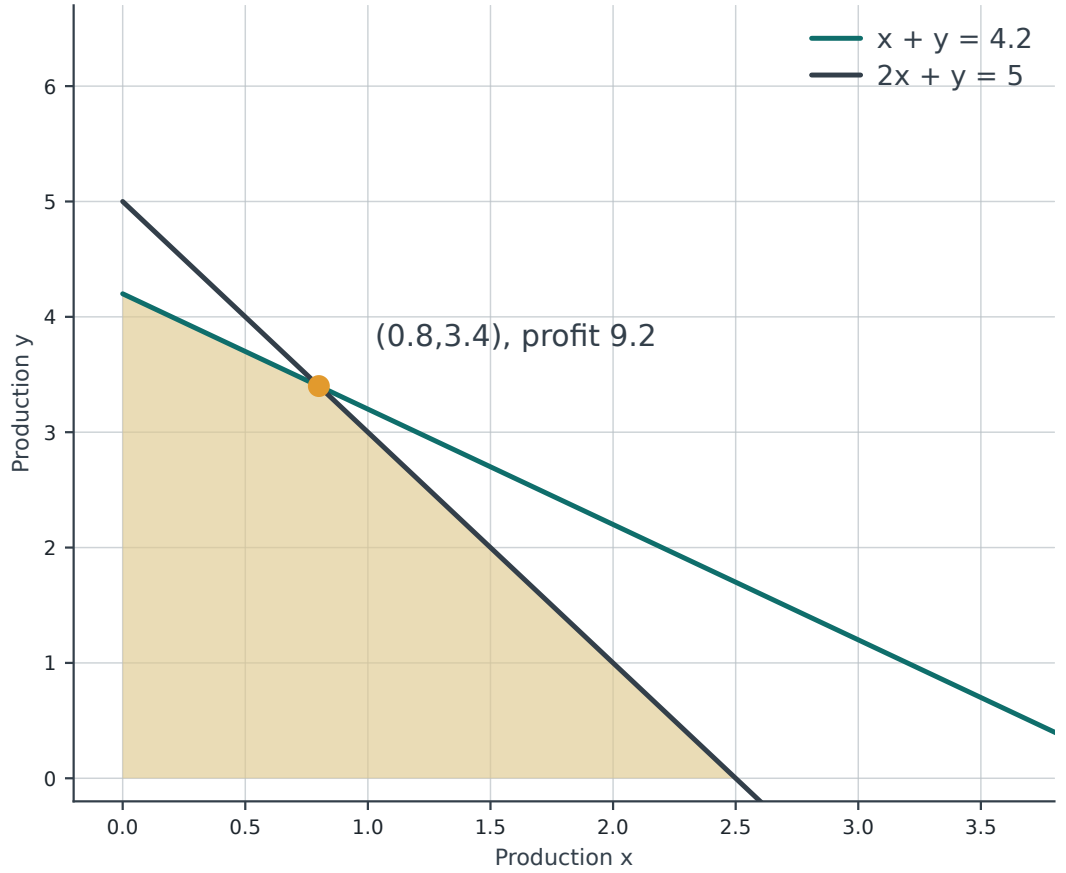


Inside the interval: $b_1 = 4.2$

$x = 0.8;\ y = 3.4;\ z = 9.2$

$\Delta b_1 = 0.2;\ \Delta z = 1 \times 0.2 = 0.2$

The shadow-price prediction is exact here.

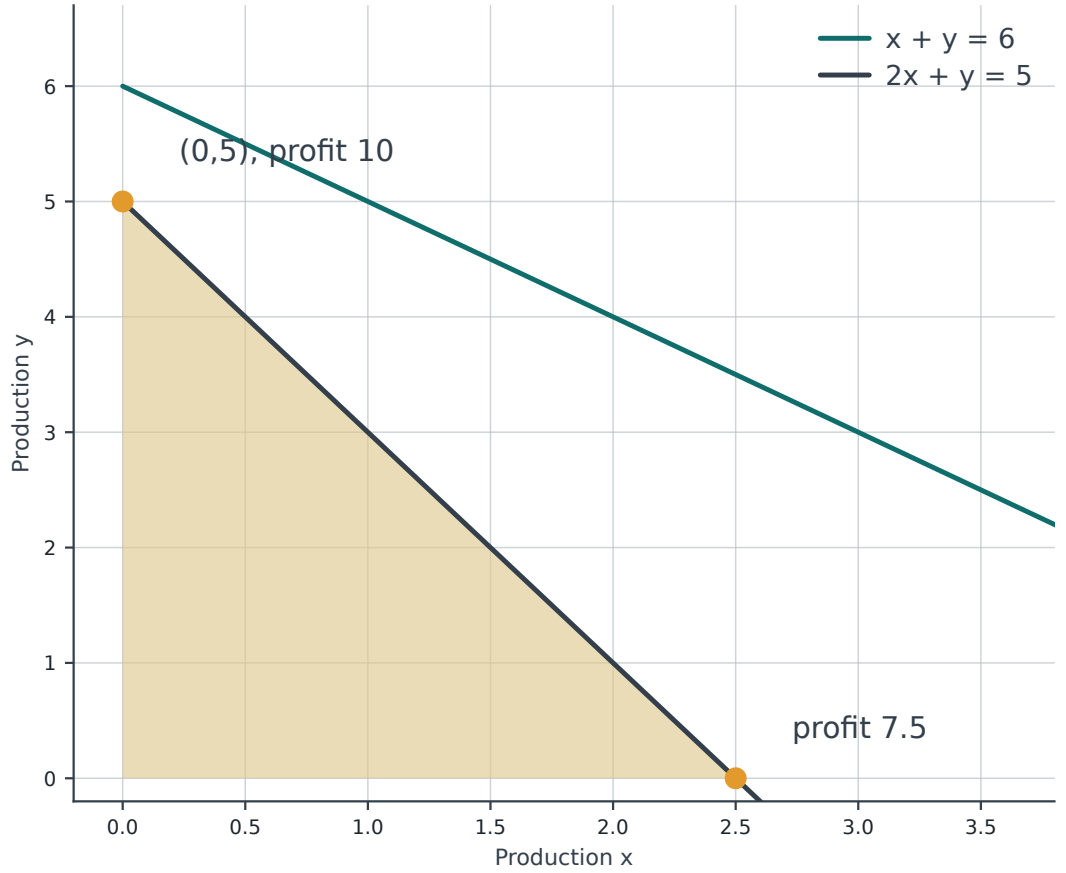


Outside the interval: $b_1 = 6$

Old intersection: $(-1, 7)$, infeasible.

Old prediction: $9 + 1(6 - 4) = 11$.

Reoptimize: $(x, y) = (0, 5)$, $z = 10$.

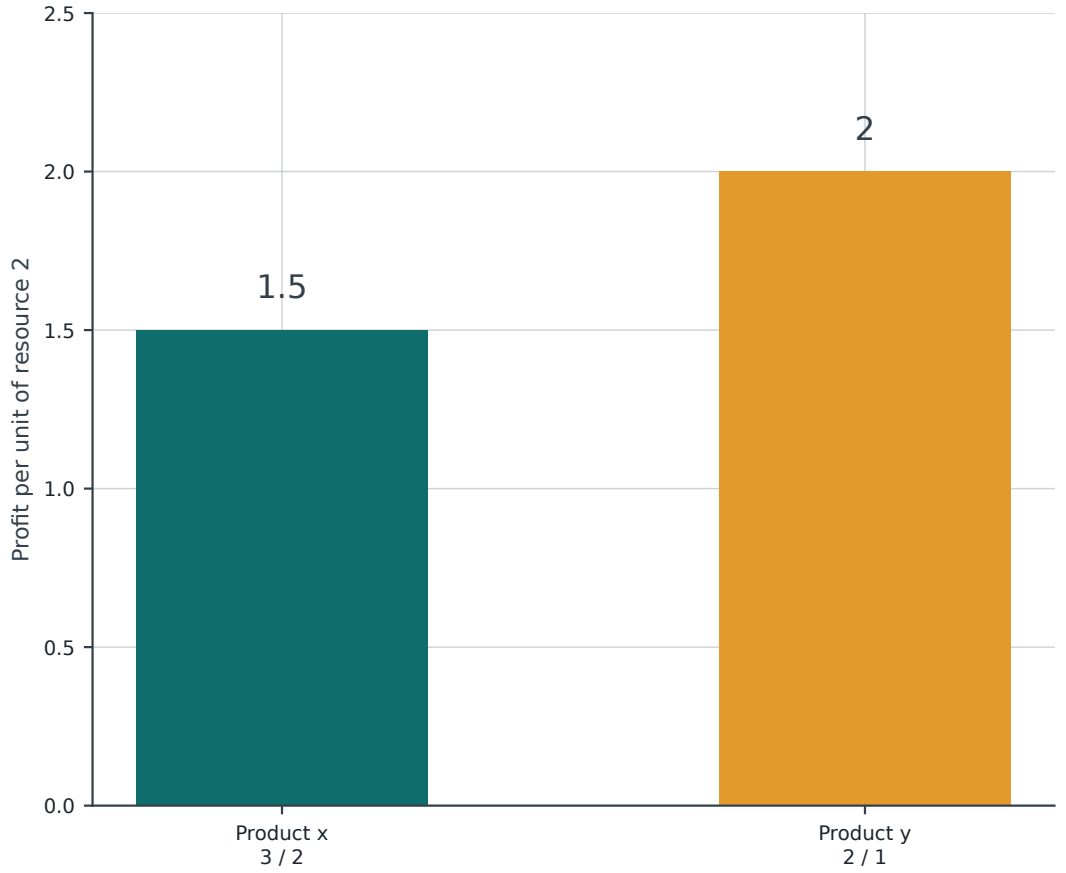


Why producing y gives the larger profit

For $b_1 = 6$: $y \le 5 - 2x$ for every feasible plan.

$z = 3x + 2y \le 10 - x \le 10$

Feasible (x,y) = (0,5) attains 10.

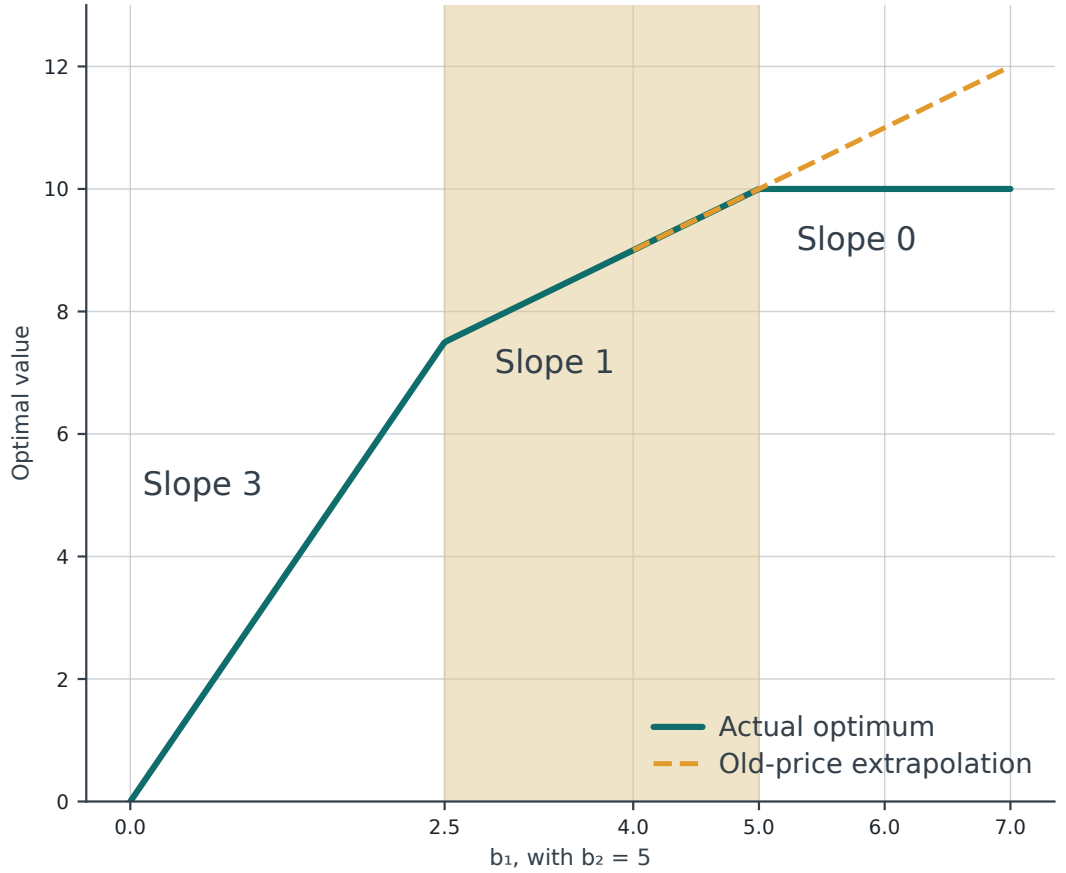


Shadow prices are slopes of the value function

For $b_1 \geq 0$: $z^* = \min(3b_1, b_1 + 5, 10)$

Slopes: 3, then 1, then 0

Use the slope only within its valid segment.

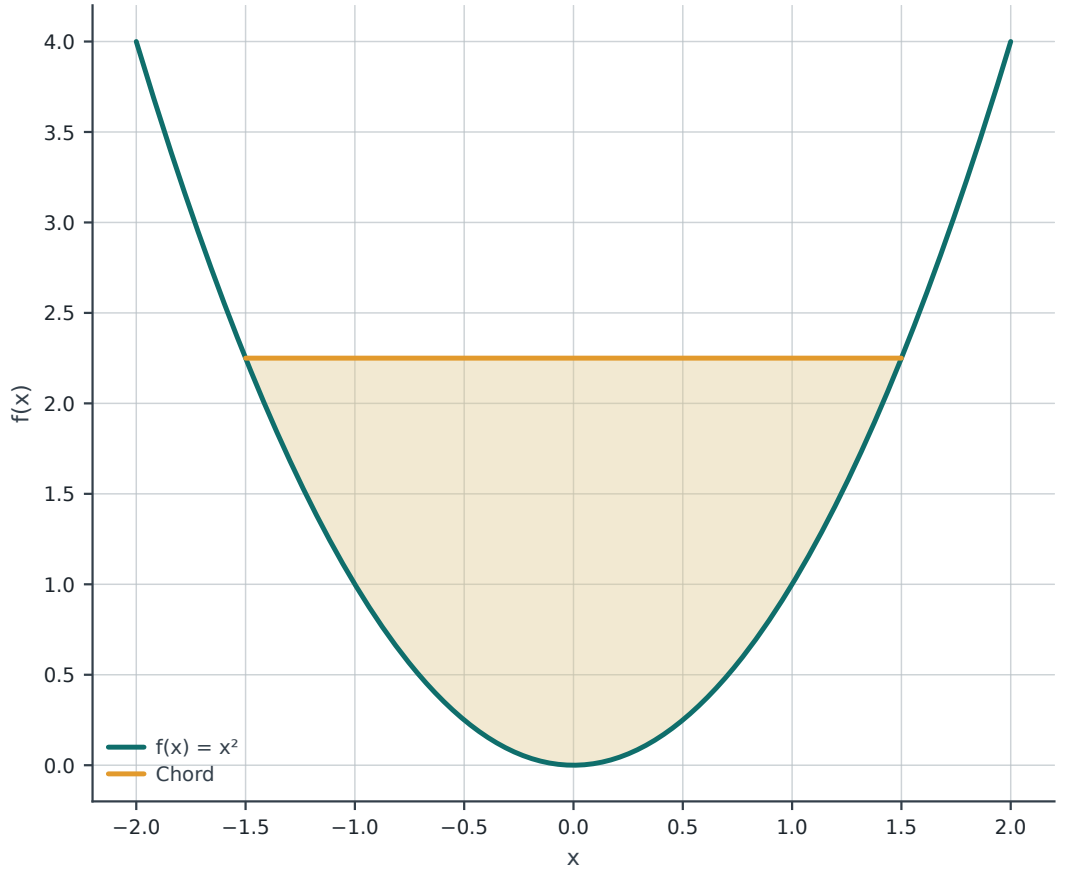


Convexity requires a domain-wide check

Assume f is C^2 on an open convex domain.

Convex ⇔ Hessian is positive semidefinite everywhere.

A Hessian check at one point is not enough.

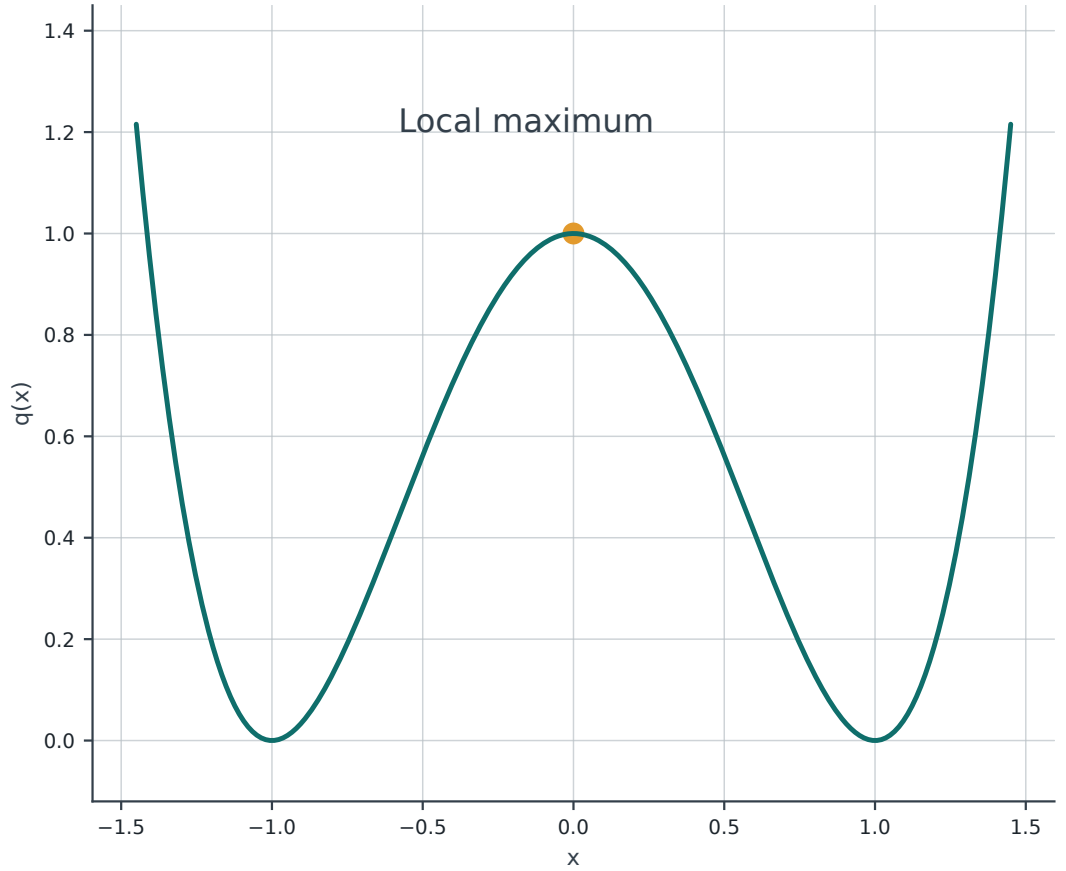


A stationary point may be a maximum

$q(x) = (x^2 - 1)^2$

$q'(x) = 4x(x^2 - 1)$; $q''(x) = 12x^2 - 4$

At $x = 0$: $q' = 0$, $q'' = -4 < 0$.

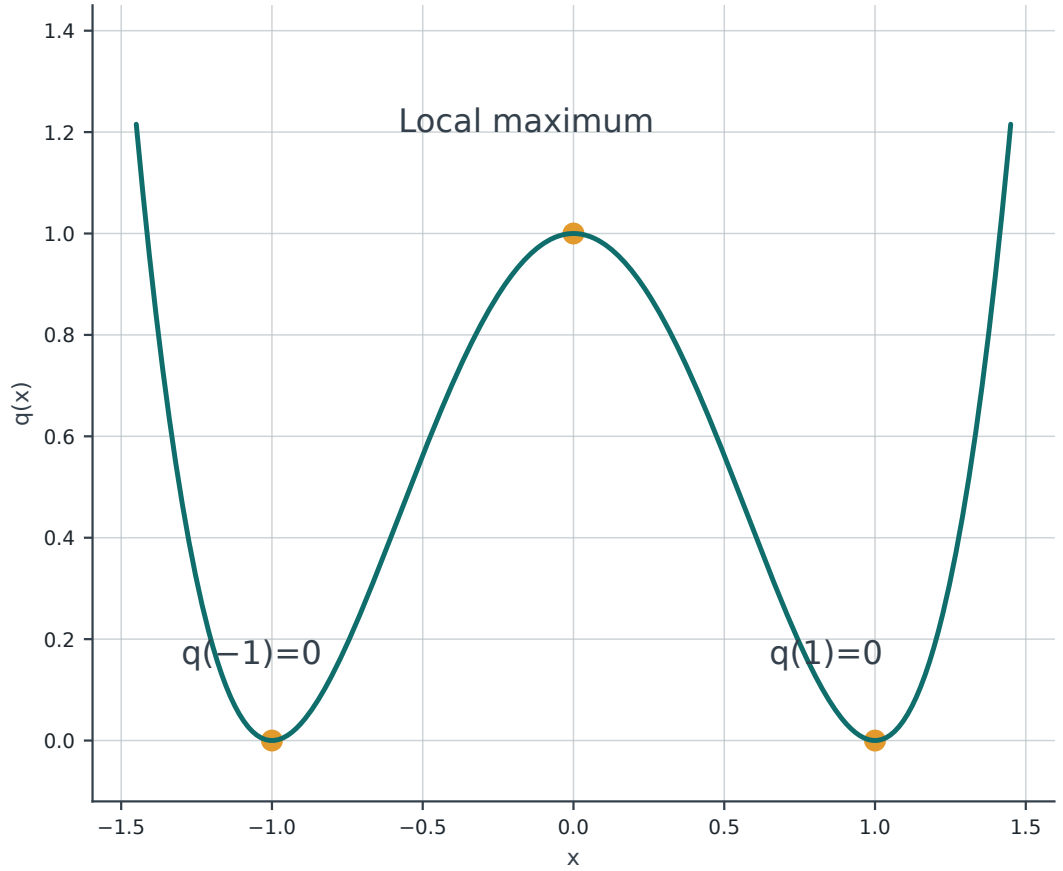


Check values and classify the point

$q(0) = 1;\ q(-1) = q(1) = 0$

$q(x) \geq 0$ for every real x.

Thus $x = \pm 1$ are global minima; $x = 0$ is a local maximum.

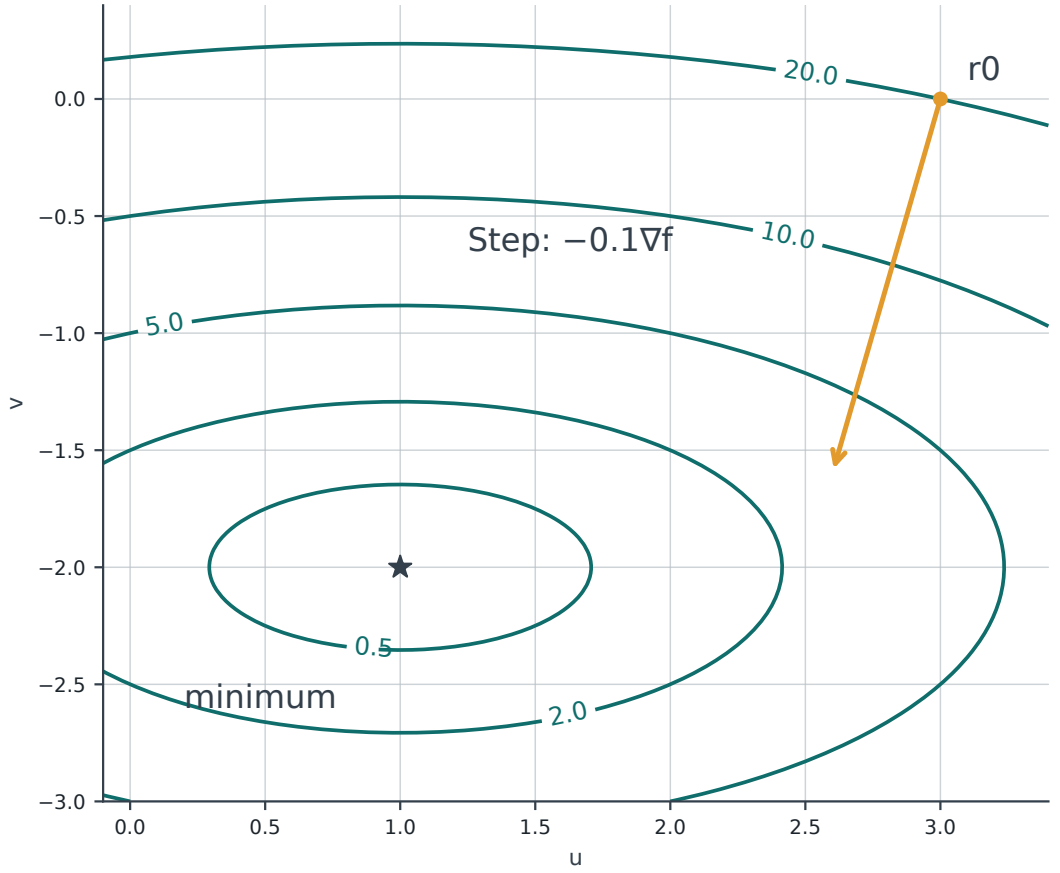


Gradient descent: compute the gradient

NLP coordinates u,v: $f = (u-1)^2 + 4(v+2)^2$

$\nabla f = (2(u-1), 8(v+2))$

At (3,0), $\nabla f = (4,16)$ and $f = 20$.

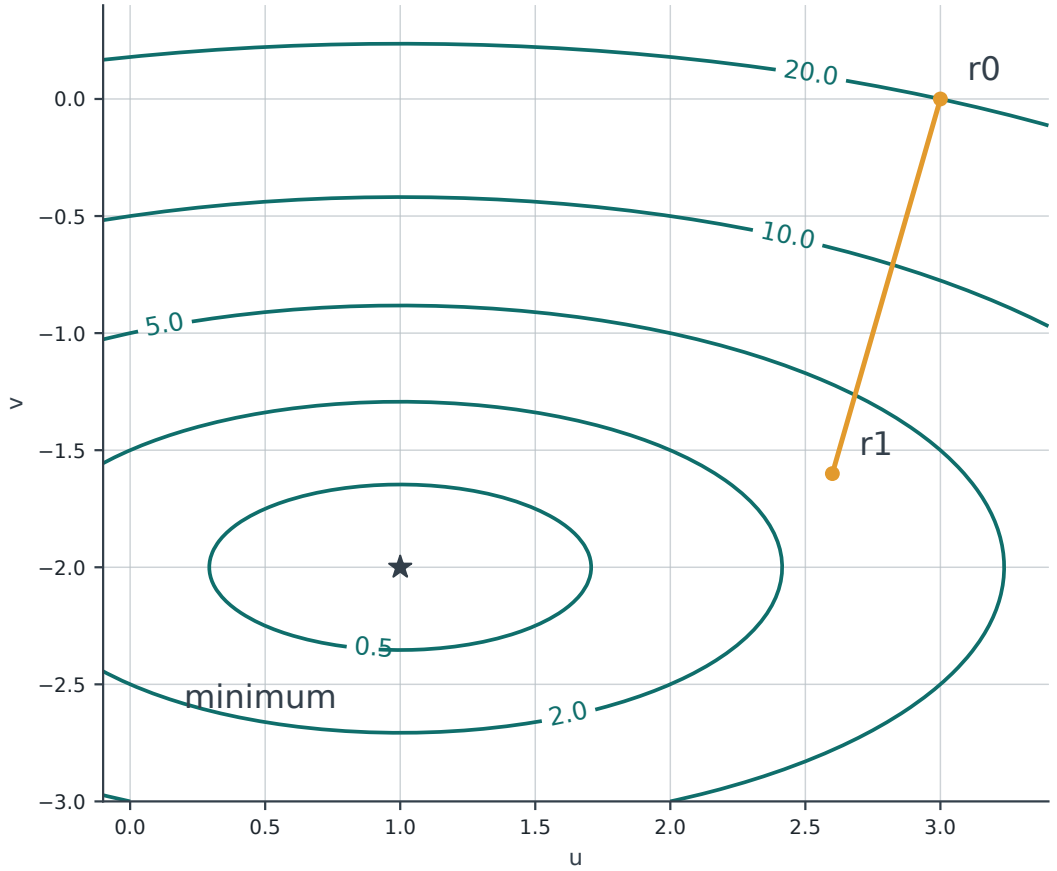


One gradient step is not the optimum

Let $r_k = (u_k, v_k)$. Use $\alpha = 0.1$: $r_1 = r_0 - \alpha \nabla f(r_0)$.

$(u_1, v_1) = (3,0) - 0.1(4,16) = (2.6, -1.6)$

$f(r_1) = 1.6^2 + 4(0.4)^2 = 3.2$

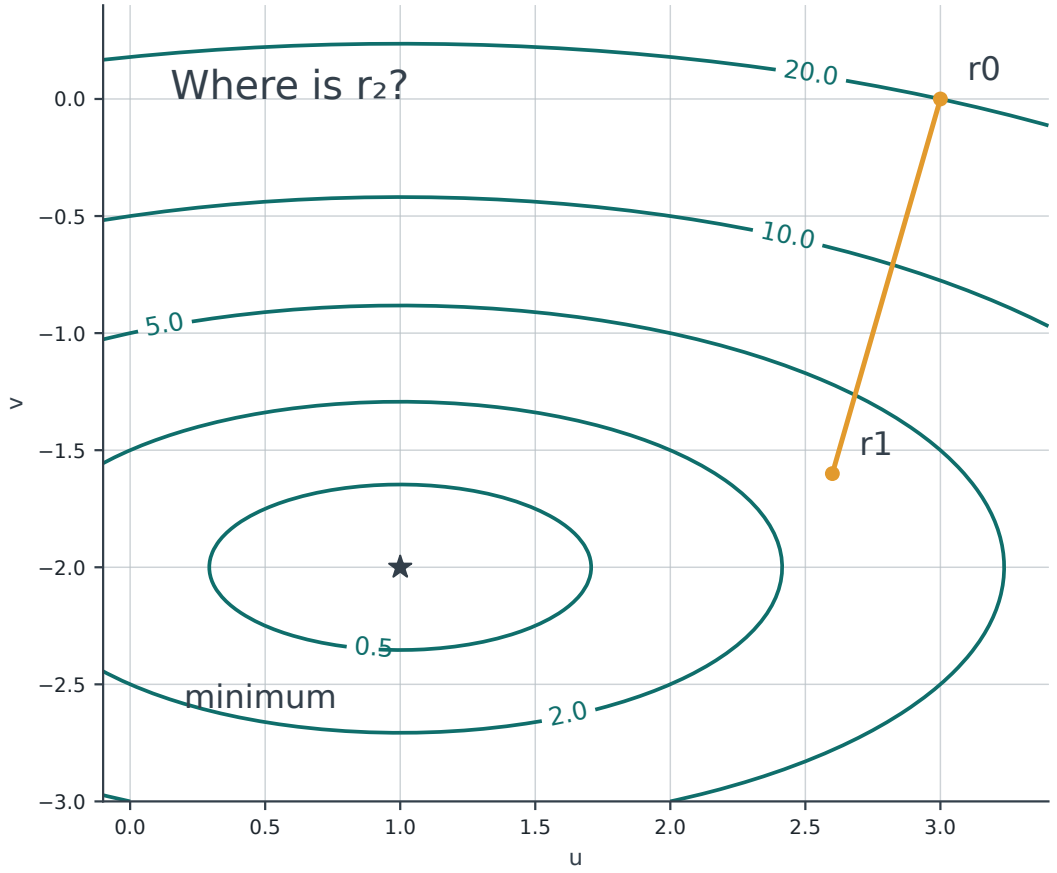


Practice: compute the second step

Start now at (2.6, −1.6); keep α = 0.1.

Compute the new gradient, point, and objective.

Does reusing the old gradient give the same method?

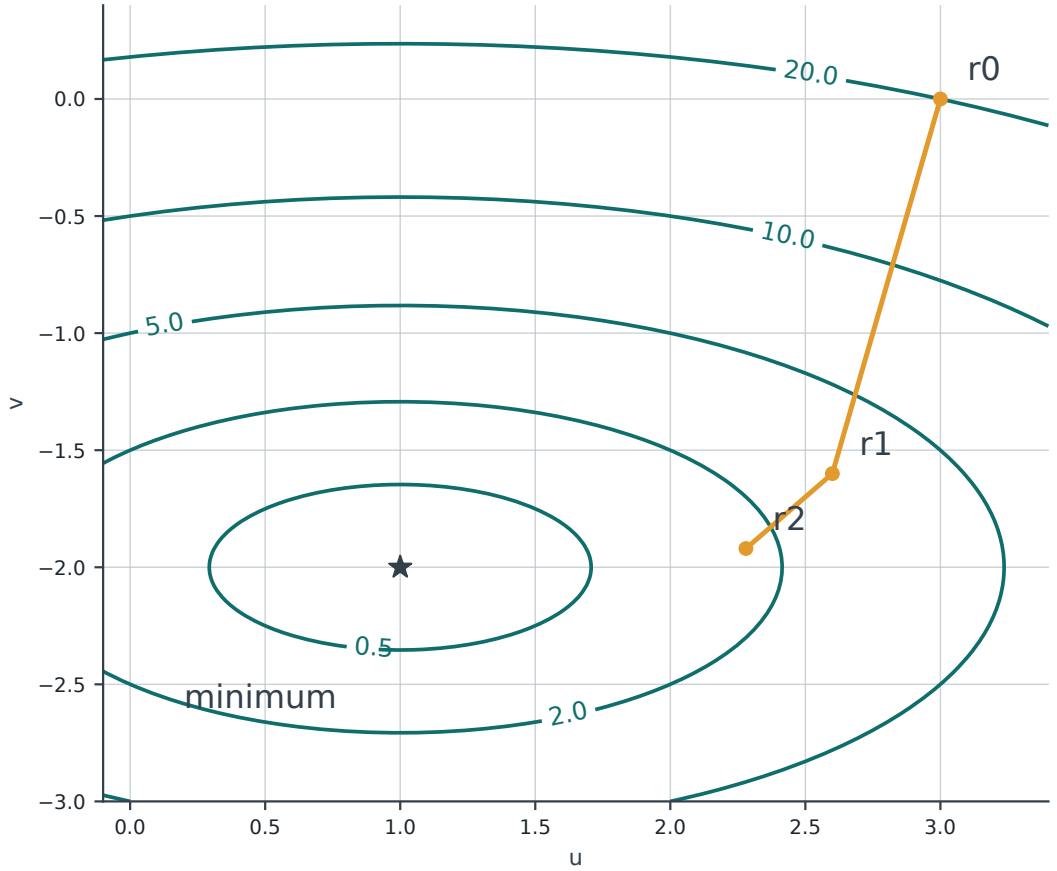


Solution: update the gradient every time

$\nabla f(2.6,-1.6) = (3.2,3.2)$

$r_2 = (2.6,-1.6) - 0.1(3.2,3.2) = (2.28,-1.92)$

$f(r_2) = 1.28^2 + 4(0.08)^2 = 1.664$

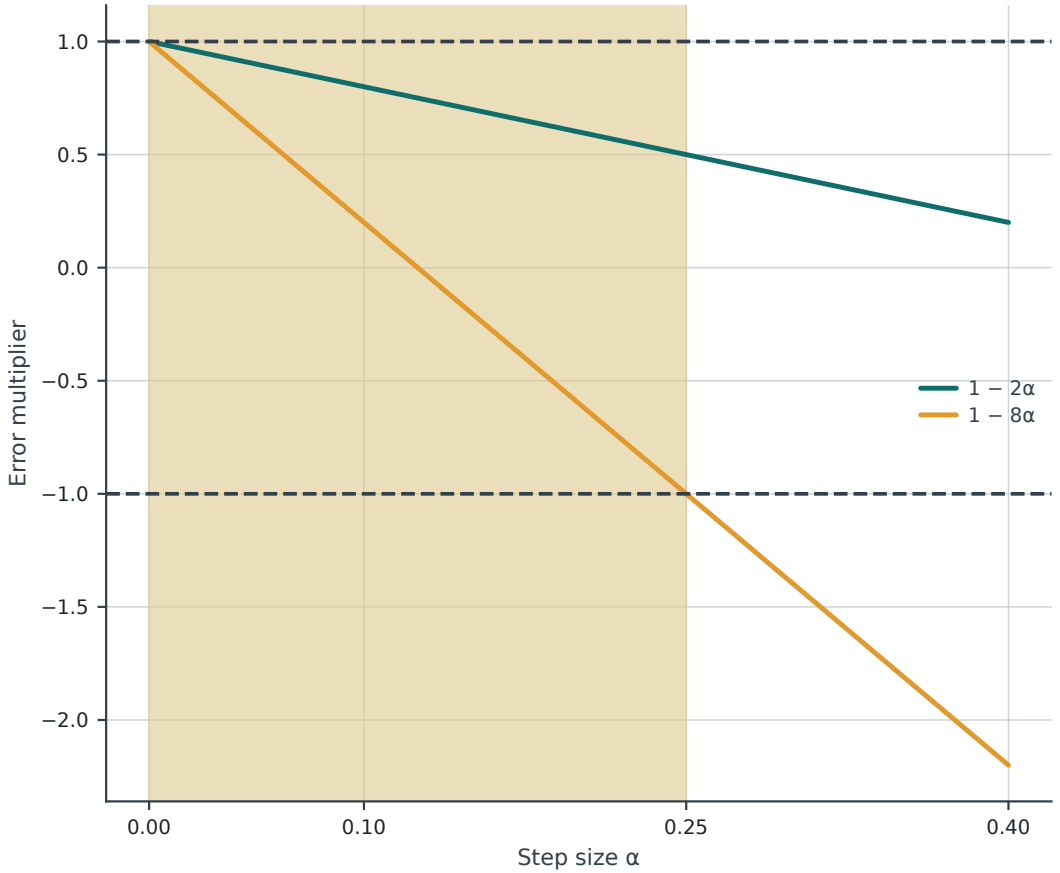


Derive the constant-step convergence range

Let $e_u = u-1$ and $e_v = v+2$.

$e_{u,\text{new}} = (1-2\alpha)e_u$; $e_{v,\text{new}} = (1-8\alpha)e_v$

Both shrink for every start when $0 < \alpha < 0.25$.

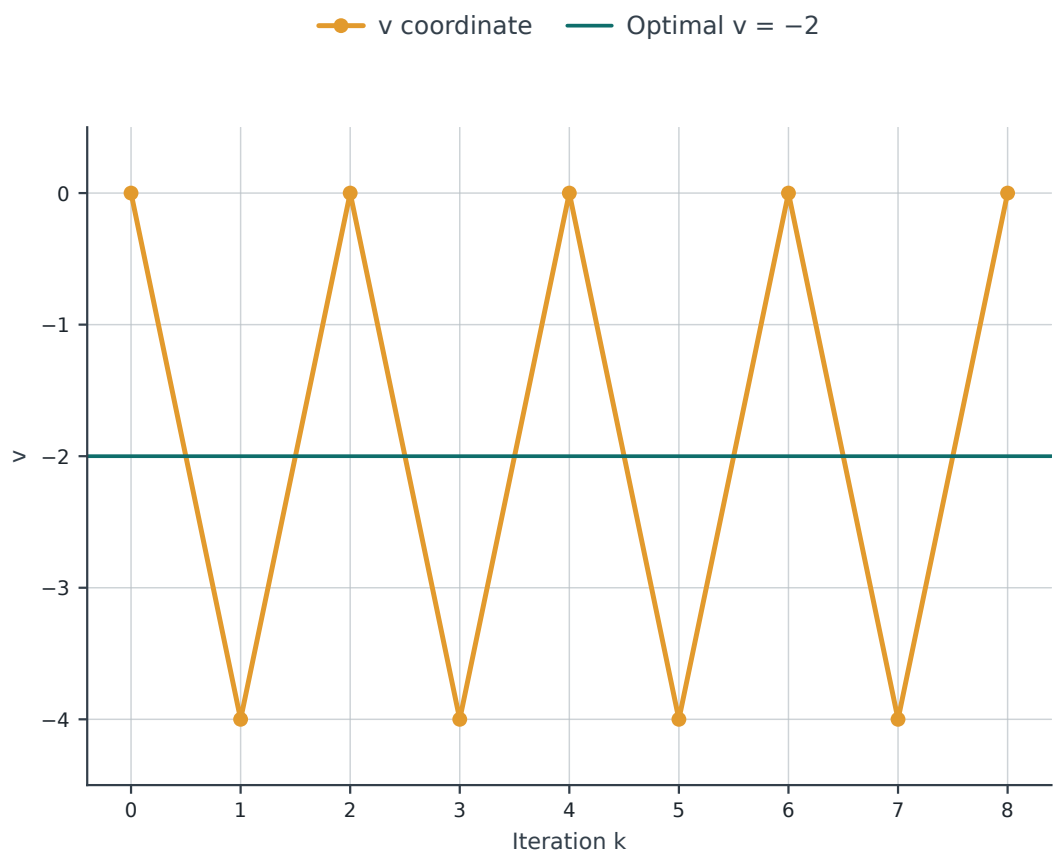


At α = 0.25, one coordinate oscillates

e_u multiplier = 0.5; e_v multiplier = −1.

From (3,0), v alternates: 0, −4, 0, −4, …

$f(r_k) = 4(1/4)^k + 16 \to 16$, not 0.

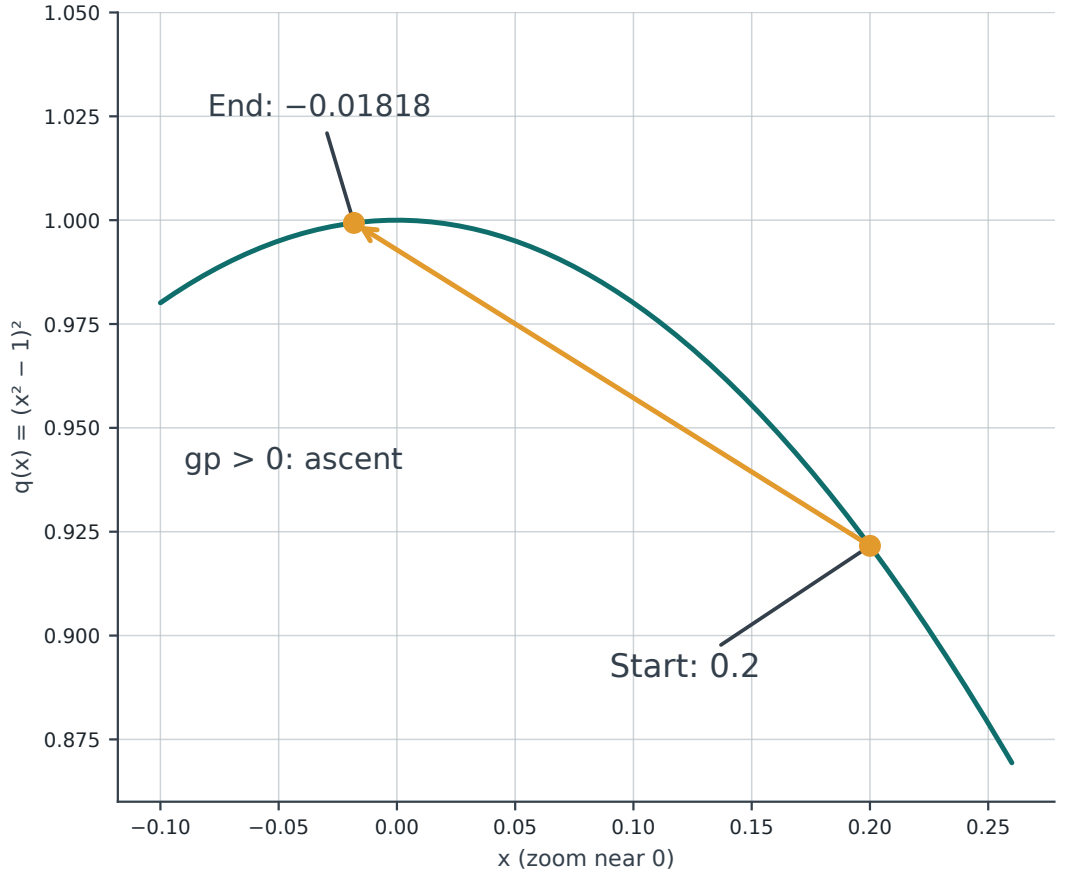


Newton directions also need a check

$p = -g/H$ in one dimension.

At $x = 0.2$ for q: $g = -0.768$, $H = -3.52$.

$p \approx -0.21818$; full step $x+p \approx -0.01818$.

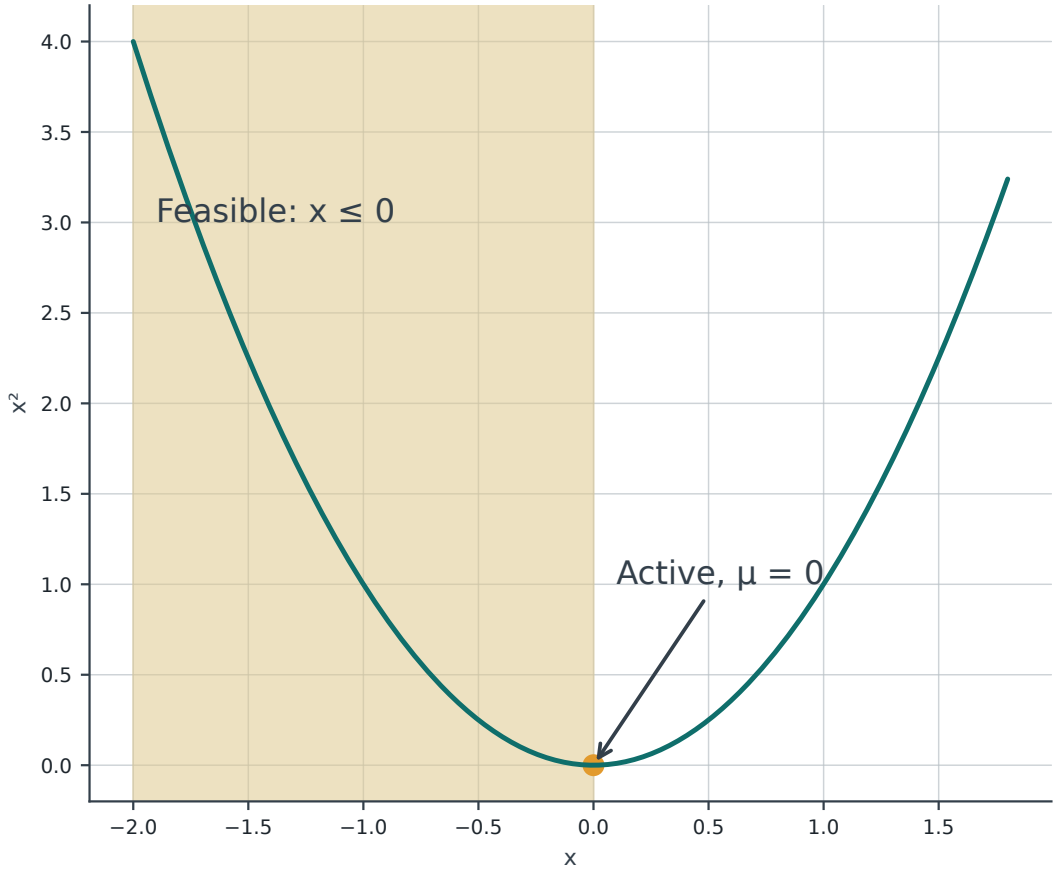


KKT: active does not mean positive multiplier

Minimize x^2 subject to $x \leq 0$.

At $x^* = 0$: the constraint is active, $\mu = 0$.

$2x + \mu = 0$; $\mu \geq 0$; $\mu x = 0$.



Bring your reasoning to onsite class

Monday, November 9, 2026: attend onsite.

Bring your LP and NLP calculations and questions.

Review the steps, not just the final answers.

